2/13-11

Arrow Smith }
Place } ODE

$\{\mathbb{R}^2\} \to \mathbb{R}^n \xrightarrow{?} L_2$ $\mathbb{R}^n : \{[x_1 \quad , x_n]^T$

## ifferenhal equations

$\bar{x}' = \bar{F}(\bar{x},t)(*)$, $\bar{x} \in \mathbb{R}^n$, $\bar{F} \in \mathbb{R}^n$, $t \in [a,b] \subset \mathbb{R}$

$?(t) = ?$ $?\ [a,b]\ ?$ Existence of solutions:

If $\bar{F}(\bar{x},t)$ is continuous in the domain $D \times I$, where D - open domain in $\mathbb{R}^n$

I - open interval in $\mathbb{R}$

$\bar{x}_0 \in D$; $t_0 \in I$

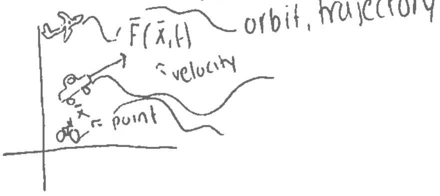
Then there is a (smaller) interval $]t_0-\varepsilon, t_0+\varepsilon[$ and there is $\bar{x}(t)$ defined on $]t_0-\varepsilon, t_0+\varepsilon[$ satisfying $(*)$

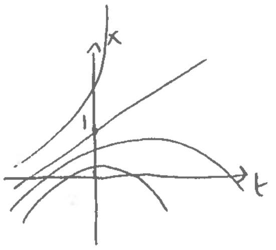
Remember that it might be several solutions going through the same point.

Pr. 1.1.2. in the book (through one point there can go only one orbit)

If $F(\bar{x},t)$ and $\frac{\partial F(x,t)}{\partial x_i}$ are both continuous, then the existing solution will be unique.

Phase space





$C>0 \Rightarrow x$ monotone in time

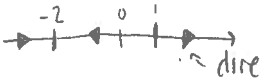
$C=0 \Rightarrow$ straight line

$x' = x - t \Rightarrow x(t) = 1 + t + Ce^t$

arbitrary constant

Stationary points are zeroes of $F(x,t)$

$x' = (x-1)(x+2)$



$\Rightarrow -2$ and $1$ are stationary points.

These two stationary points are very different. Point $x_i$ is an unstable stationary point.
Point $x_2$ is a "stable" stationary point

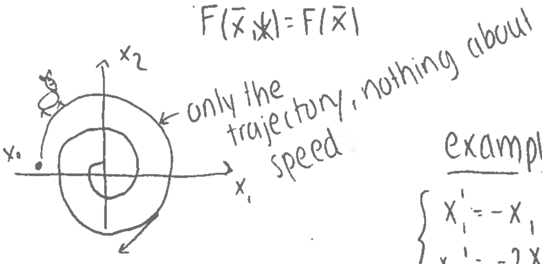
one can try to count how many qualitatively different configurations might exist for a 1-dim system with two stationary points?

$x' = (x^2-1)\cdot 2$ Try!

Antonomous systems in plane 1.3

$F(\bar{x},\cancel{t}) = F(\bar{x})$

hase-plane



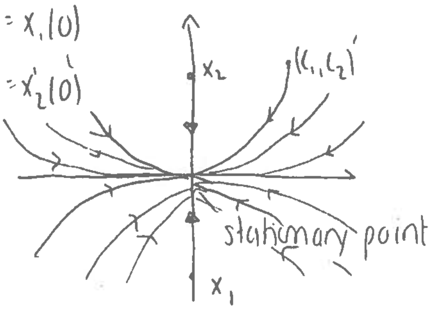
Phase portrait of the system
(representative orbits and stationary points)

examples

$$\begin{cases} x_1' = -x_1 \\ x_2' = -2x_2 \end{cases} \Rightarrow \begin{cases} x_1 = C_1e^{-t} \\ x_2 = C_2e^{-2t} \\ \to 0 \text{ faster, } t \to \infty. \end{cases}$$

$C_1 = x_1(0)$

$C_2 = x_2'(0)$



[illegible]

ex 1.4.1

$x_1' = x_2'$

$x_2' = -x_1$

$\bar{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \bar{x}$ in vector form.

$i \sim \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

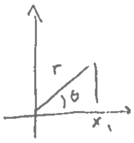
introduce polar coordinates

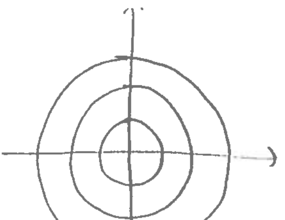
$\begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{cases}$ $r = r(t)$ $\theta = \theta(t)$ $r^2 = x_1^2 + x_2^2$

$\tan\theta = \frac{x_2}{\ } = \frac{\sin\theta}{\ }$

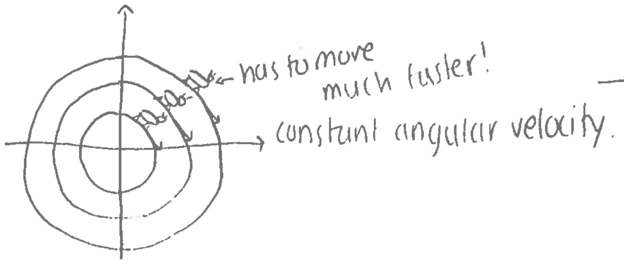
$(r^2) = 2rr' \equiv (x_1 + x_2) = 2(x_1 x_1' + x_2 x_2') = 2(x_1 x_2 - x_2 x_1) = 0$     $r' = 0 \rightarrow r(t) = r(0)$

$(\tan\theta)' = \frac{1}{\cos^2\theta} \cdot \theta' = \left(\frac{x_2}{x_1}\right)' = \frac{x_2' x_1 - x_1' x_2}{x_1^2} = \frac{-x_1^2 - x_2^2}{x_1^2} = -\frac{r^2}{x_1^2} = -\frac{1}{\cos^2\theta}$





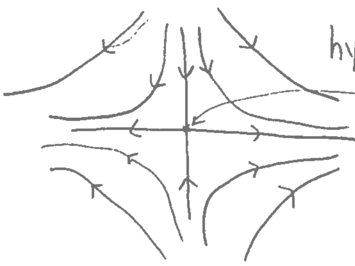
$\Rightarrow \theta' = -1 \Rightarrow \theta = \theta_0 - t$



ex

$\begin{cases} x_1' = x_1 \\ x_2' = -x_2 \end{cases} = \begin{cases} x_1 = x_1(0)e^{t} \\ x_2 = x_2(0)e^{-t} \end{cases}$   trick not formal $\begin{cases} \frac{dx_1}{dt} = x_1 \\ \frac{dx_2}{dt} = -x_2 \end{cases}$   $\frac{dx_1}{dx_2} = -\frac{x_1}{x_2}$   $\int \frac{dx_1}{x_1} = -\int \frac{dx_2}{x_2} \Rightarrow \ln x_1 = -\ln x_2 + C$

$\ln x_1 + \ln x_2 = C \Leftrightarrow \ln x_1 \cdot x_2 = C \Rightarrow x_1 \cdot x_2 = \text{constant}$



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1 june, examination.

Phase plane, stationary points, "stability", orbits. (last time, important in many problems)

Law of mass action in chemistry

$A + B \xrightarrow{k} C$

two types of particles. molecules, animals etc.)

new particle (chemical reaction)

A means also the amount of particles of type A.

characterizes how active the reaction is.

$\frac{dA}{dt} = -k(A \cdot B)$ , $\frac{dB}{dt} = -k(A \cdot B)$ , $\frac{dC}{dt} = k(A \cdot B)$

natural with multiplication follows from probability theory, more likely for two particles to merge when there are many of them.

valid for large numbers of particles.

Bimolecular reaction (most of reactions in chemistry are of this type)

similar models can be used for animals competing in nature

ex) Foxes and rabbits

$\frac{dR}{dt} = k_b \cdot R \cdot R - k_d \cdot R - k_{kill} F \cdot R$
birth   disease

$\frac{dF}{dt} = \ldots$

Aids spreading

- healthy
- infected, but not ill
- $ill_1$
- $ill_2$
- $ill_3$

Should add time and space dependence.

---

$A \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} X$

$B + X \xrightarrow{k_2} Y + D$

$2X + Y \xrightarrow{k_3} 3X$

$\begin{cases} \frac{dX}{dt} = k_1 A - k_{-1} X + k_3 X^2 Y \\ \frac{dY}{dt} = k_2 B - k_3 X^2 Y \end{cases}$

Many constants!
changing variables lets to make this number smaller.

$u = \sigma X$ , $v = \sigma Y$ , $\sigma = \sqrt{k_3/k_{-1}}$ , $\tau = k_{-1} t$

$a = \frac{k_1}{k_{-1}} \sqrt{k_3/k_{-1}}\, A$ , $b = \sqrt{k_2/k_{-1}}\, B$

non-dimensional variables

$\frac{du}{dt} = ⓐ - u + u^2 v$

$\frac{dv}{dt} = ⓑ - u^2 v$

Stationary states:

$u^0 = a+b$

$v^0 = \frac{b}{(a+b)^2}$

$$\begin{cases} a-u+u^2v=0 \\ b-u^2v=0 \end{cases}$$

$\boxed{\text{stable or not?}}$

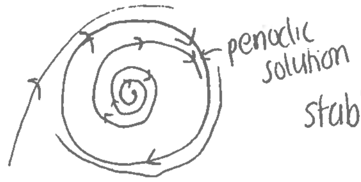
Linearised system around $(u^0, v^0)$ (can only say things in a close neighbourhoud to the stationary point

$$\left\{ \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{b-a}{b+a} & (a+b)^2 \\ \frac{-2b}{b+a} & -(a+b)^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right.$$

Matrix
eigenvalues
eigenvectors.

Conclusions; stability

Periodic solution



stable if spiral moving inwards.

Large systems which don't deviate very much from a stable position

Stochastic models

$P(n_x, n_y, t)$ - distribution function

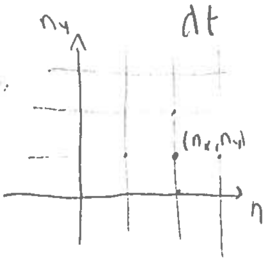
number of x particles

number of y particles

Probability to find $n_x$ x particles and $n_y$ y particles.

Master equation

$$\frac{dP(n_x,n_y,t)}{dt} = -\left[k_1 n_A + k_{-1} n_x + k_3 n_x(n_x-1)n_y\right] P(n_x,n_y) + k_1 n_A P(n_{x-1}, n_y) k_2 n_B P(n_x, n_{y-1}$$



Large system of ODE

Death and birth systems

MASTER EQUATION

$P(n_{x+1}, n_y) - P(n_x, n_y) \approx \frac{\partial P}{\partial x}(x,y)$ can approximate the above system with this.

$\Rightarrow$ PDE $\quad \frac{\partial P}{\partial t} - \Delta P - V \nabla P$

PROJECT

xamples of projects:

1. Freezing a lake (Stefan problem), nonlinear because of moving boundary condition, the ice (melting or both)

2. Describe completely living of an electric lamp

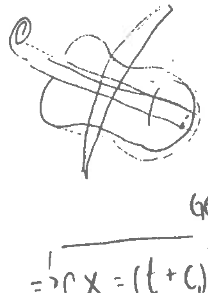
3. Aids or other sickness spreading.

4. Animal fight. (Both deterministic and stochastic)

. Traffic dynamics.

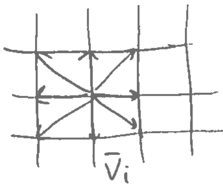
. Modeling flows or diffusion by Lattice Boltzman equation

. Dynamics of a music instrument



. Reaction-diffusion and pattern ormation.

1. Translation (jump)



$f(v_i, x, t)$

$\rho = \sum f_i$

$V = \sum f_i v_i$

2. $\frac{\partial f_i}{\partial t} = (f^{\rho v} - f_i)\frac{\rho}{\tau}$

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$\dot{x}_1 = 3x_1^{2/3}$

$\dot{x}_2 = 1$

velocities
velocity vector.

$\dot{x}_1 = \frac{dx_1}{dt}$

$\frac{1}{3}\int \frac{dx}{x^{2/3}} = \int dt \Rightarrow x_1^{1/3} = t + C \Rightarrow$

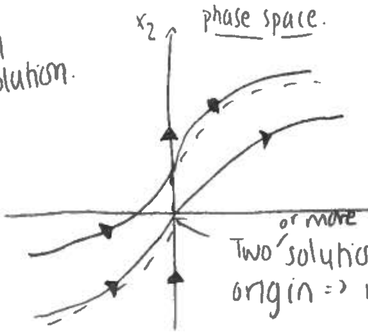
General solution.

$$\boxed{\begin{cases} x_1 = (t+C_1)^3 \\ x_2 = t + C_2 \end{cases}}$$

one special solution $\begin{cases} x_1 = 0 \\ x_2 = t + C_2 \end{cases}$

If the derivative velocity vector isn't a smooth function, uniqueness might be lost

phase space.



Two or more solutions through the origin $\Rightarrow$ not uniqueness soclines (making a sketch of the phase plane without a solution to the system) for $x = F(\bar{x})$

lines where $F_2 / F_1$ = constant.

x 1.4.5

$_1 = x_1^2$

$_2 = x_2(2x_1 - x_2)$

1. The origin $(0,0) = (x_1, x_2)$ is a stationary point.
2. The system is symmetrical with respect to the $x_2$-axis..
3. $\frac{dx_2}{dx_1} = \frac{x_2(2x_1 - x_2)}{x_1^2} = C$

withem - Linear and nonlinear waves.

$x_2(2x_1 - x_2) = Cx_1^2$

$x_1^2 - (x_1 - x_2)^2 = Cx_1^2 \iff (1-C)x_1^2 = (x_1 - x_2)^2 \Rightarrow \underline{C \leq 1}$ !!!

$\boxed{\text{For } x_1 > 0}$ $\pm\sqrt{1-C}\,x_1 = (x_1 - x_2) \Rightarrow x_2 = x_1(1 \pm \sqrt{1-C})$ All isoclines are straight lines!

$=0;\ x_2 = 0\ ;\ x_2 = 2x_1$

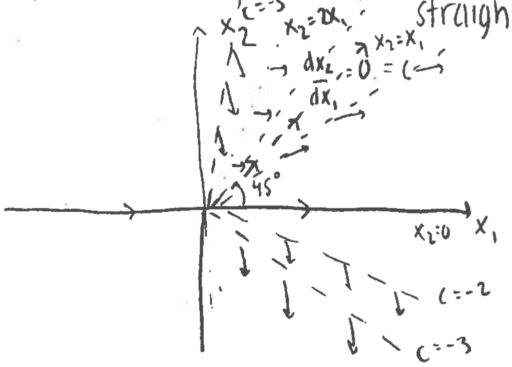
$= 1 \Rightarrow x_2 = x_1$

$= \frac{1}{2}\quad x_2 = x_1(1 + \sqrt{\tfrac{1}{2}})$

$x_2 = x_1(1 - \sqrt{\tfrac{1}{2}})$

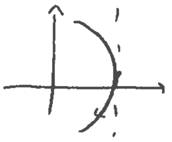
$= -3;\ x_2 = 3x_1;\ x_2 = -x_1$

$= -2\ ;\ x_2 = (1 \pm \sqrt{3})x_1$

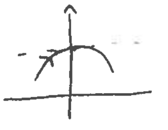


where $\frac{F_2}{F_1}$ = constant.

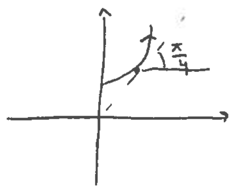
$x_1' = F_1 = 0$

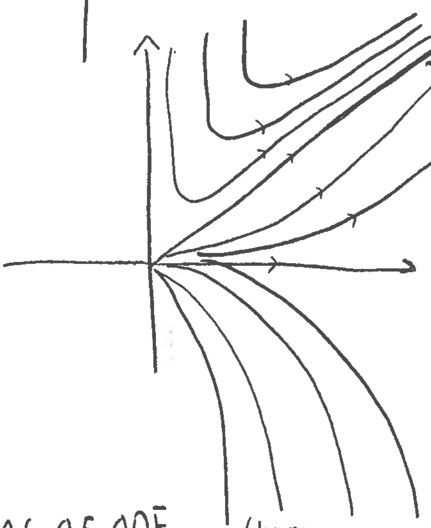


$x_2' = F_2 = 0$



$\frac{F_2}{F_1} = 1$





Vision on rapport

tudy a new topic in ysics, biology, chemistry etc...

) Find yourself or in literature a mathematical model.

) Find a numerical lgorithm to solve 2).

) Implementing.

) Analysing the results.

) Oral report.

uning 2-6, write a eport.

Structure.

Like a scientific article. LaTex

1) Description of the physical model
2) Descr. of mathematical model and it's properties.
3) Numerics, code
4) Results.
5) Conclusions.

LINEAR SYSTEMS OF ODE (homogeneous)

$\bar{x}' = A\bar{x}$ A $n \times n$-matrix, $\bar{x} \in \mathbb{R}^n$

Change of variables

$\bar{x} = M\bar{y} \leftarrow$ new variable $\qquad \bar{y} = M^{-1}\bar{x}$

$\uparrow$ invertible $n \times n$-matrix.

$\uparrow$ to have uniqueness (bijection)

$\bar{x}' = AM\bar{y}$

$M\bar{y}' = AM\bar{y} \Rightarrow \bar{y}' = M^{-1}AM\bar{y}$

The same problem in terms of new variables.

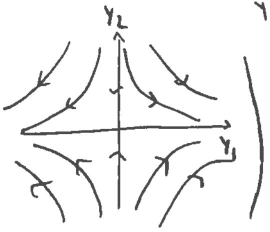
Matrices A and $M^{-1}AM$ are equivalent (similar). In particular A and $M^{-1}AM$ have the same eigenvalues.

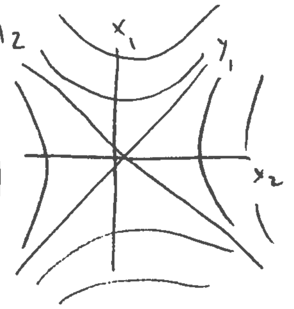
igenvalues to A are solutions to the equation $\det(A - \lambda I) = 0 \quad (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) = 0.$

igenvectors corresponding to $\lambda$ $\{\lambda_i\}_{i=1}^{n}$ are vectors $\bar{v}_i$ satisfying $A\bar{v}_i - \lambda_i\bar{v}_i = 0$

x) $\begin{cases} x_1' = x_2 \\ x_2' = x_1 \end{cases}$ $\quad \begin{matrix} (x_1 + x_2)' = x_1 + x_2 \\ (x_1 - x_2)' = -(x_1 - x_2) \end{matrix}$ $\quad \begin{cases} y_1 = x_1 + x_2 \\ y_2 = x_1 - x_2 \end{cases} \Rightarrow \begin{cases} y_1' = y_1 \\ y_2' = -y_2 \end{cases}$

$\bar{y} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\bar{x}$





## Proposition 2.2.1

$(\in \mathbb{R}^{2\times 2})$

et A be a 2×2-matrix. Then there is a real matrix M such that $\underbrace{M^{-1}AM}_{J}$ has one of the ollowing forms:

**JORDAN FORM OF MATRIX A**

) $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $\lambda_1 > \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$

c) $\begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix}$ $\lambda_0 \in \mathbb{R}$

) $\begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{bmatrix}$ $\lambda_0 \in \mathbb{R}$

d) $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$, $\beta > 0$; $\alpha, \beta \in \mathbb{R}$

If A is real, complex eigenvalues come in conjugated pairs!

roof:

) If A has two distinct eigenvalues: $\lambda_1, \lambda_2$, then taking corresponding eigenvector $\bar{v}_1, \bar{v}_2$ we define $M = [\bar{v}_1 \vdots \bar{v}_2]$ $AM = [A\bar{v}_1 \vdots A\bar{v}_2] = [\lambda_1\bar{v}_1 \vdots \lambda_2\bar{v}_2] = M\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $\Rightarrow M^{-1}AM = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

using the fact that eigenvectors corresponding to different eigenvalues have to be linearly independent

) If A is diagonal and has two equal eigenvalues.

) If A is not diagonal and has two similar eigenvalues $\lambda_1 = \lambda_2 = \lambda_0 \in \mathbb{R}$. $\xrightarrow[\text{eigenvector}]{\text{corresponding}}$ $\bar{u}_0 = \bar{m}_1$

ake $M = [\bar{m}_1 \vdots \bar{m}_2]$. $AM = [A\bar{m}_1 \vdots A\bar{m}_2] = [\lambda_1\bar{m}_1 \vdots A\bar{m}_2] = M\left[\lambda_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} \vdots M^{-1}A\bar{m}_2\right] = M\begin{bmatrix} \lambda_0 & \vdots & ? \\ 0 & \vdots & \end{bmatrix}$

$\hat{}$ some vector not parallell to $\bar{m}_1$ $\Rightarrow$ invertible matrix.

$\underset{\text{n-singular}}{\hat{\text{ı}}}$ $^{-1}AM = \begin{bmatrix} \lambda_0 & \vdots & c \\ 0 & \vdots & \lambda_0 \end{bmatrix}$ Because $M^{-1}AM$ is diagonal, and such a matrix has the eigenvalues on the diagon

$M_1 = M\begin{bmatrix} 1 & \vdots & 0 \\ 0 & \vdots & c^{-1} \end{bmatrix}$ $M_1^{-1}AM_1 = \begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix}$

### 28/3-11 canonical forms of matrices

d) $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \cdot \begin{bmatrix} \lambda & -\gamma \\ \gamma & \lambda \end{bmatrix} = ?$ $\left.\begin{matrix} \lambda_1 = \alpha + i\beta \\ \lambda_2 = \alpha - i\beta \end{matrix}\right\}$ eigenvalues to the matrix A. M has to satisfy: $\exists M : M^{-1}AM = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ $AM = M\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$

$(\alpha + i\beta)(\lambda + i\gamma)$

$M = [m_1 \vdots m_2]$. $[A\bar{m}_1 \vdots A\bar{m}_2] = [\alpha\bar{m}_1 \vdots \alpha\bar{m}_2] + [\beta\bar{m}_2 \vdots -\beta\bar{m}_1] = [\alpha\bar{m}_1 + \beta\bar{m}_2 \vdots \alpha\bar{m}_2 - \beta\bar{m}_1] =$

$= [A\bar{m}_1 \vdots A\bar{m}_2] \Leftrightarrow [(A - \alpha I)\bar{m}_1 - \beta I\bar{m}_2 \vdots \beta I\bar{m}_1 + (A - \alpha I)\bar{m}_2] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\underbrace{\begin{bmatrix} A - \alpha I & -\beta I \\ \beta I & A - \alpha I \end{bmatrix}}_{P} \begin{bmatrix} \bar{m}_1 \\ \bar{m}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

← complex conjugate.

$Q = \begin{bmatrix} A - \alpha I & \beta I \\ -\beta I & A - \alpha I \end{bmatrix}$ $PQ = \begin{bmatrix} [(A-\alpha I)^2 + \beta^2 I] & 0 \\ 0 & [(A-\alpha I)^2 + \beta^2 I] \end{bmatrix} =$

$[A^2 - 2\alpha AI + (\alpha^2 + \beta^2)I] = P(A)$ ← polynomial

$\underbrace{P(\lambda) = \lambda^2 - 2\alpha\lambda + (\alpha^2 + \beta^2)}$ characteristic polynomial to A

$\lambda_1 = \alpha + i\beta$
$\lambda_2 = \alpha - i\beta$

$\boxed{\begin{matrix} 2\lambda = tr(A) \\ \alpha^2 + \beta^2 = det(A) \end{matrix}} \Rightarrow$

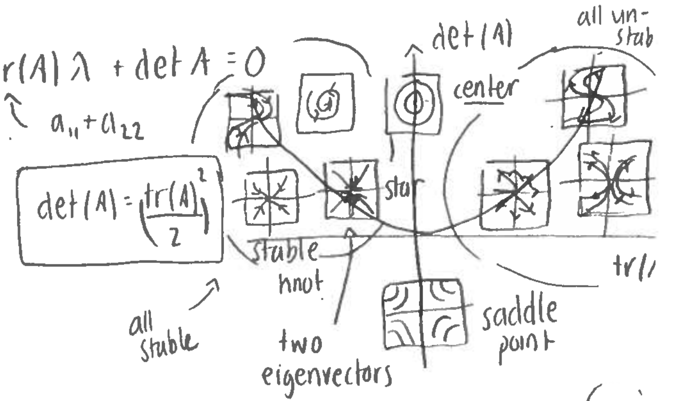
P(A) = 0 according to Hamilton Cayley theorem wanted to solve $\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\begin{bmatrix} m_1 \\ \bar{m}_2 \end{bmatrix} = \begin{bmatrix} a_{11} \dots \\ \alpha_{22} \\ -\beta \\ 0 \end{bmatrix} \Rightarrow M = \begin{bmatrix} a_{11}-\alpha & -\beta \\ a_{22} & 0 \end{bmatrix}$

---

$M^{-1}AM = J$ can always be found s.t. $J$ has got one of the canonical forms described above depending on eigenvalues we get different canonical forms $J$ and different types of <u>phase ortrait</u>.

$\lambda_1, \lambda_2$ are zeroes of polynomial $\det \begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix} = \lambda^2 - \underset{a_{11}+a_{22}}{\operatorname{tr}(A)}\lambda + \det A = 0$

$$\lambda_{1,2} = \frac{\operatorname{tr}(A)}{2} \pm \sqrt{\frac{\operatorname{tr}(A)^2}{4} - \det(A)}$$

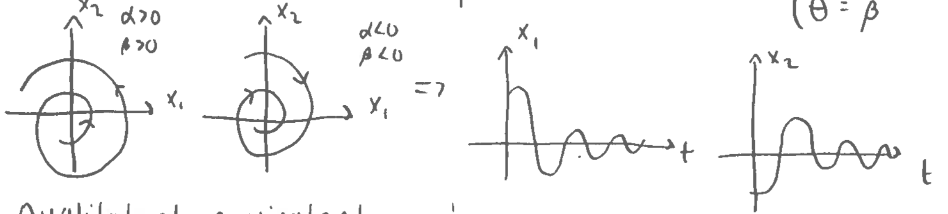


$A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$ $\overset{\nearrow}{\,}s^2.$ $x_1' = \alpha x_1 - \beta x_2$ $x_2' = \beta x_1 + \alpha x_2$ $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ did before.

$x_1 = r\cos\theta$ $x_2 = r\sin\theta$ $r = r(t)$ $\theta = \theta(t)$ $r^2 = x_1^2 + x_2^2$ $\tan\theta = \frac{x_2}{x_1}$ differentiate with respect to t. $\Rightarrow \begin{cases} r' = \alpha r \\ \theta' = \beta \end{cases} \Rightarrow r = r_0 e^{\alpha r}$ $r_0 = r(0)$

$\theta = \theta_0 + \beta t$ $\theta_0 = \theta(0)$

↖ the sign of $\beta$ gives spirals in positive/negative direction



<u>Qualitatively equivalent</u>

$\exists$ continuous invertible mapping from one phase portrait to another.

<u>Evolution operator</u>

$\dot{x} = F(x)$ $(\bar{x}' = \bar{F}(\bar{x}))$ ; $\bar{x}(0) = \bar{x}_0$

$\varphi_t(\bar{x}_0) = \bar{x}(t)$

$\boxed{x_0 \xrightarrow{\varphi_t} x(t)}$

$\boxed{\varphi_t(\varphi_s) = \varphi_{t+s}}$

$\bar{x}' = \underset{\text{matrix}}{A}\bar{x}$ $x' = \underset{\text{number}}{a}x$ ; $x(0) = x_0$

$\varphi_t = ?$ ? $x = x_0 e^{at}$

$\varphi_t(\bar{x}_0) = \boxed{e^{At}} x_0$ $e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^k}{n!}$

$(e^{At})' = A \sum_{n=0}^{\infty} \frac{A^n t^k}{n!}$ $(e^{At})' = (I + At + \frac{A^2t^2}{2} + \dots)' = A + \frac{\cancel{2}At}{\cancel{2}} + \frac{\cancel{3}At^2}{\cancel{3}\cdot 2} + \dots$

$(e^{At}\bar{x}_0)' = A(e^{At}\bar{x}_0)$ $\bar{x} = e^{At}\bar{x}_0 \Rightarrow \boxed{\bar{x}' = A\bar{x}}$ $\left( \begin{matrix} J = M^{-1}AM \\ A = MJM^{-1} \end{matrix} \Rightarrow e^{At} = e^{(MJM^{-1})t} \right)$

$\bar{x}' = A\bar{x}$, $\bar{x} = M\bar{y}(t) = Me^{Jt}\bar{y}(0) =$

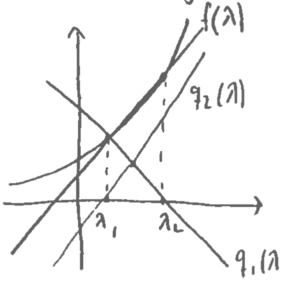
$= (Me^{Jt}M^{-1})\bar{x}_0$ $\varphi_t = \underline{\underset{\nwarrow 2}{M}e^{Jt}M^{-1} = e^{At}}$

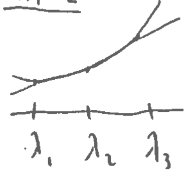
<u>Method by Sylvester</u>

If A has two distinct eigenvalues <u>$\lambda_1, \lambda_2$</u>

Introduce matrix $Q_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2}$, $Q_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1}$

$\frac{\lambda - \lambda_2}{\lambda_1 - \lambda_2} = q_1(\lambda)$ $\frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1} = q_2(\lambda)$





$f(\lambda) \approx f(\lambda_1)q_1(\lambda) + f(\lambda_2) \cdot q_2(\lambda)$ → linear interpolation polynomial.

$f(A) = f(\lambda)Q_1(A) + f(\lambda_2) \cdot Q_2(A)$

$Q_1Q_2 = Q_2Q_1 = 0$

$Q_1^2 = Q_1$, $Q_2^2 = Q_2$ $\Rightarrow A^n = (\lambda_1 Q_1 + \lambda_2 Q_2)^n = \lambda_1^n Q_1 + \lambda_2^n Q_2$

dynamiskt system - mat. modell. En variabels värde ändras med tiden enligt en regel som bara bero
av värden modellen själv har skapat. Fix regel bestämmer tidsberoendet hos en punkt i rummet.

phase-space - represents all possible states of a system. in a plot
(phase plane in 2-D)

state space / phase space
autonomous equation
phase portrait
qualitative equivalence
steepest descent.

(*) $\dot{x} = \mathbb{X}(x)$ autonomous eq. Since $\dot{x}$ determined by x itself.

← Real-valued.

$\dot{x}(t) = \mathbb{X}(t, x(t))$ $x(t) = \xi(t)$ Solves (*)

phase portrait : geometrisk beskrivning av banorna för ett dynamiskt system

phase portrait: plot of typical trajectories in the phase space.

Two different equations of the form $\dot{x} = \mathbb{X}(x)$ are qualitatively equivalent if they have the same nbr of fixed point, arranged in the same order and of the same nature along the phase line:

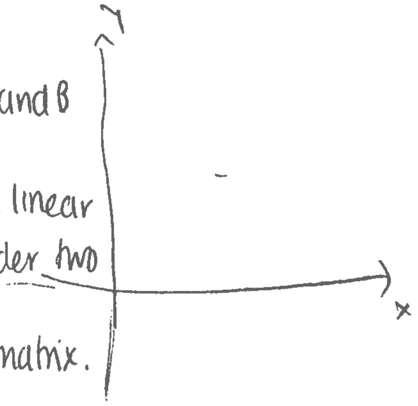
$\dot{\mathbf{x}} = \mathbb{X}(\mathbf{x})$ linear if $\mathbb{X} : \mathbb{R}^n \to \mathbb{R}^n$ linear mapping.

$\mathbb{X}(\mathbf{x}) = A\mathbf{x}$
↖ coefficient matrix

$\dot{\mathbf{x}} = A\mathbf{x} = AM\mathbf{y} =$

$A = \underset{A}{\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}}$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overset{M}{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$B = P^{-1}AP$, then A and B are similar.
eepresent the same linear transformation under two different bases.
P change-of-basis matrix.



$\frac{dy}{dx} = y' = 1$

$y = x$

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n \cdot t^n}{n!} = \sum_{n=0}^{\infty} \left(\frac{\lambda_1^n t^n}{n!}\right) Q_1 + \left(\frac{\lambda_2^n t^n}{n!}\right) Q_2 = e^{\lambda_1 t} Q_1 + e^{\lambda_2 t} Q_2$$

If A has multiple eigenvalue $\lambda_1 = \lambda_2 = \lambda_0$ : $Q = A - \lambda_0 I$ $\quad A^n = (\lambda_0 I + Q)^n = \lambda_0^n I + n\lambda^{n-1} Q$ , $Q^2 = 0$ for $n \geq 2$

$$\Rightarrow e^{At} = \sum_{k=0}^{\infty} \left(\frac{\lambda_0^n}{n!} I + \frac{n\lambda_0^{n-1}}{n!} Q\right) t^n = e^{\lambda_0 t}(I + t(A - \lambda_0 I))$$

9/3-1! Affine systems

$'=A\bar{x}$ - homogeneous

$'=A\bar{x}+\bar{h}(t)$ Affine system. Use the evolution operator. for homogeneous problem

$_t(\bar{x}_0) = \bar{x}(t) = e^{A(t-t_0)}\bar{x}_0$ $\quad \underbrace{e^{-At}}_{\text{inversion of } e^{At}}(\bar{x}') = e^{-At}A\bar{x} + e^{-At}\bar{h}(t)$ $\quad \int_{t_0}^{t} \frac{d}{dt}\{e^{-At}\bar{x}\} = \int_{t_0}^{t} e^{-At}\bar{h}(t)$ $\Rightarrow e^{-At}\bar{x} - e^{-At_0}\underset{\bar{x}(t_0)}{\underset{\shortparallel}{\bar{x}_0}} =$

$$= \int_{t_0}^{t} e^{-As} h(s)ds \ \overset{\text{multiply from the left with } e^{At}}{\Rightarrow} \ \bar{x} = \underbrace{e^{A(t-t_0)}\bar{x}_0}_{\text{solution to the homogeneous equation.}} + \int_{t_0}^{t} e^{A(t-s)} h(s)ds.$$

Examples

$y_1' = \lambda_0 y_1 + y_2$ $\quad y_1 = (C_1 + tC_2)e^{\lambda_0 t}$

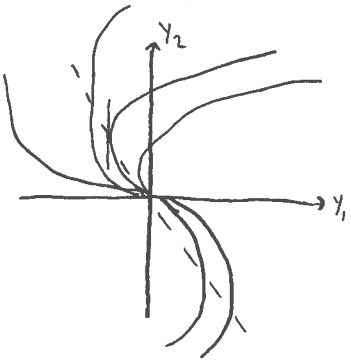
$y_2' = \lambda_0 y_2$ $\quad y_2 = e^{\lambda_0 t} C_2$

$\lambda_0 > 0$.

$$\begin{vmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{vmatrix}$$

isoclines $y_1' = 0$

$y_2 = -\lambda_0 y_1$



General phase portraits (Transformation M)

$\bar{x}' = A\bar{x}$ ; $\bar{x} = M\bar{y} = [\bar{m}_1 \vdots \bar{m}_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y_1\bar{m}_1 + y_2\bar{m}_2$

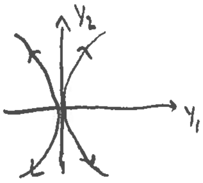
new variable — symmetry axises.

$A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ $\quad \lambda_1, \lambda_2$? $\bar{x}' = A\bar{x}$

$\det(A - \lambda I) = 0$.

$\lambda^2 - tr(A)\lambda + \det(A) = 0$ $\quad \lambda^2 - 6\lambda + 7 = 0 \Rightarrow \lambda_{1,2} = 3 \pm \sqrt{2} \approx 1.4.$ $\quad \lambda_1 = 3 + \sqrt{2} > 0$. $\quad \lambda_2 = 3 - \sqrt{2} > 0$. | unstable knot

both positive



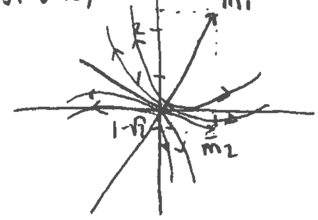
$(A - \lambda_1 I)\bar{m}_1 = 0$

$(A - \lambda_2 I)\bar{m}_2 = 0$ eigenvectors.

$(2 - 3 - \sqrt{2})m_1^{(1)} + 1 - m_1^{(2)} = 0$

$1 \cdot m_1^{(1)} + (4 - 3 - \sqrt{2})m_1^{(2)} = 0$ $\quad \Rightarrow \frac{m_1^{(1)}}{m_1^{(2)}} = 1 + \sqrt{2}$ $\quad \bar{m}_1 = \begin{bmatrix} 1 \\ 1+\sqrt{2} \end{bmatrix}$

equation for $\bar{m}_2$ (first one) $\quad (2 - 3 + \sqrt{2})m_2^{(1)} + m_2^{(2)} = 0 \Rightarrow \frac{m_2^{(2)}}{m_2^{(1)}} = 1 - \sqrt{2} \approx -0.4$



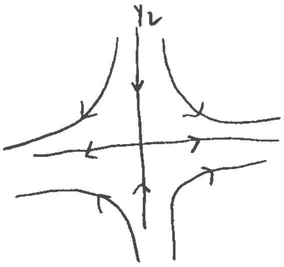
$\lambda^2 - tr(A)\lambda + \det(A)$

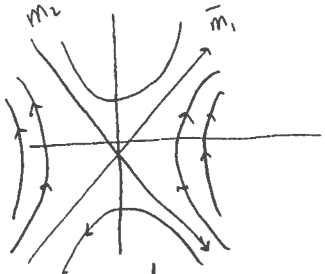
$\lambda_{1,2} = \frac{tr(A)}{2} \pm \sqrt{\frac{tr(A)^2}{4} - \det(A)}$ $\quad A = \begin{vmatrix} -3 & 4 \\ 4 & -2 \end{vmatrix}$ $\quad \frac{tr(A)}{2}^2 = \frac{25}{4}$

$\lambda_{1,2} = -\frac{5}{2} \pm \sqrt{\frac{25}{4} + 10} = \frac{-5 \pm \sqrt{65}}{2}$ $\quad \lambda_1 = \frac{-5 + \sqrt{65}}{2}$ , $\lambda_2 = \frac{-5 - \sqrt{65}}{2}$ $\quad \frac{m_2^{(2)}}{m_2^{(1)}} = (3 + \frac{-\sqrt{65} - 5}{2})$ $\approx -\frac{3}{4}$ $<($

$(-3 - \lambda_1)m_1^{(1)} + 4m_1^{(2)} = 0$

$(-3 - (-\frac{5 + \sqrt{65}}{2})m_1^{(1)} + 4m_1^{(2)} = 0$ $\quad \frac{m_1^{(2)}}{m_1^{(1)}} = (3 + \frac{\sqrt{65} - 5}{2})\frac{1}{4} > 1$ slightly





$x'' + 2x' + 2x = u(t)$

i) Transform to a system

$x_1 = x$ $\qquad x_1' = x_2$

$x_2 = x'$ $\qquad x_2' + 2x_2 + 2x_1 = u(t)$ $\qquad \Leftrightarrow \bar{x}' = A\bar{x} + ut$ with $A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$

then we have $x = e^{At}x_0 + \int_0^t e^{A(t-s)}u(s)ds$, $x(0) = x_0$ $\quad e^{At} = ?$ $\quad \lambda^2 + 2\lambda + 2 = 0$ $\quad \lambda_{1,2} : -1 \pm i$ complex eigenvalues.

Sylvester's method

$$Q_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2}, \quad Q_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1}. \quad e^{At} = e^{\lambda_1 t}Q_1 + e^{\lambda_2 t}Q_2 \quad Q_1 = \begin{bmatrix} 1+i & 1 \\ -2 & -1+i \end{bmatrix}\frac{1}{2i}, \quad Q_2 = \begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix}\frac{1}{(-2i)}$$

$$e^{At} = e^{-t}\left[\frac{e^{it}}{2i}\begin{bmatrix} 1+i & 1 \\ -2 & -1+i \end{bmatrix} + \frac{e^{-it}}{(-2i)}\begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix}\right] = e^{-t}\begin{bmatrix} \cos t + \sin t & \sin t \\ -2\sin t & \cos t - \sin t \end{bmatrix}$$

31/3-11 Non-linear ODE

inearization around a fixed point

$\bar{x}' = \bar{F}(\bar{x})$ ; if $\bar{F}(\bar{0}) = \bar{0}$ - origin is a fixed point

$(\bar{x}) = \underset{\uparrow}{A\bar{x}} + g(\bar{x})$ ; $\frac{g(\bar{x})}{|\bar{x}|} \xrightarrow[|x|\to 0]{} 0$ → smaller in order than x

comment

for a fixed point $\bar{x}_0$ whe change the variables to $\bar{y} = \bar{x} - \bar{x}_0$.

acobi matrix for $\bar{F}$ in $\bar{x} = \bar{0}$

$$A = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix}\Bigg|_{\bar{x}=\bar{0}}$$

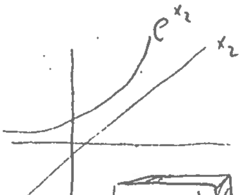
$\bar{x} = A\bar{x}$ is the LINEARIZATION of $\bar{x}' = \bar{F}(\bar{x})$ in the origin

ex) $x_1' = e^{x_1+x_2} - x_2$
$x_2' = -x_1 + x_1 x_2$

Fixed points and linearizations around

$$\begin{cases} e^{x_1+x_2} - x_2 = 0 \\ -x_1 + x_1 x_2 = 0 \end{cases} \Leftrightarrow x_1(-1+x_2) = 0 \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 1 \end{cases} \text{ or}$$

$\boxed{x_1 = 0}$ $\begin{cases} x_1 = 0 \\ e^{x_2} = x_2 \end{cases}$



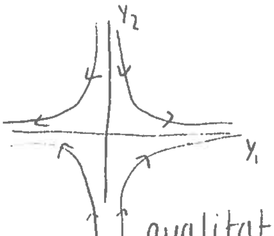
no solution. since $x_2 < e^{x_2}$

$e^{x_1+1} = 1 \Rightarrow x_1 = -1$ $\boxed{\begin{matrix} x_1 = -1 \\ x_2 = 1 \end{matrix}}$ Fixed point.

$F_1 = e^{x_1+x_2}$
$F_2 = -x_1 + x_1 x_2$

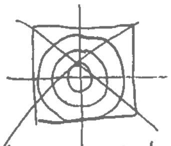
$$\Rightarrow A = \begin{bmatrix} e^{x_1+x_2} & -1+e^{x_1+x_2} \\ -1+x_2 & x_1 \end{bmatrix}\Bigg|_{\bar{x}=(-1,1)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Introduce $\begin{cases} y_1 = x_1 + 1 \\ y_2 = x_2 - 1 \end{cases}$ then $\bar{y}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \bar{y}$ Linearized system



Linearization theorem

If the linearized system has no center in the origin, then it is qualitatively equivalent to the non-linear one.



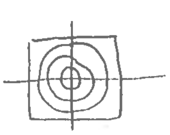
The phase portraits'll be qualitatively equivalent

SEPARATRIX is a trajectory hat approaches or emerges from a fixed point.

Tangential lines to separatrices for the system and its linearization coinside.

The local phase portrait of the non-linear system (shape and directions) will be the same.

$\begin{cases} x_1' = -x_2 + x_1(x_1^2 + x_2^2) \\ x_2' = x_1 + x_2(x_1^2 + x_2^2) \end{cases}$ the origin is a fixed point. $\xrightarrow[\text{ization}]{\text{Linear-}} \begin{cases} x_1' = -x_2 \\ x_2' = x_1 \end{cases} \Rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ - center

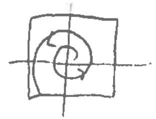


linearized system

$$\begin{cases} x_1' = -x_2 - x_1(x_1^2 + x_2^2) \\ x_2' = x_1 - x_2(x_1^2 + x_2^2) \end{cases} \begin{vmatrix} \cdot x_1 \\ \cdot x_2 \end{vmatrix} \quad \frac{1}{2}(x_1^2)' + \frac{1}{2}(x_2^2)' = -x_1x_2 + x_1x_2 - (x_1^2 + x_2^2)^2$$

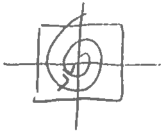
$$\frac{1}{2}(x_1^2 + x_2^2)' = -(x_1^2 + x_2^2)^2 \Rightarrow \frac{1}{2}(r^2)' = -r^4 \Rightarrow r' = -r^3_{r_0} \Rightarrow r \to 0,\ t \to \infty$$

second system



$r' = r^3 > 0$, $r \to \infty$, $t \to \infty$

first system



Home assignment N1 (two weeks for that)

1. Find a non-linear system for each type of stationary point you have learned.

1) show linearization

) Draw phase portraits for linear and non-linear ones. using Matlab (or by hand)

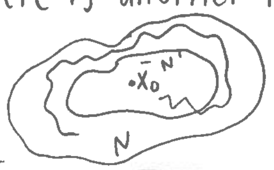
unstable ones can be found just by changin the sign

ry to find a system with two different fixed points!

## STABLE FIXED POINTS

EF: $\bar{x}' = \bar{F}(\bar{x})$ ; $\bar{x}_0$ is a fixed point, $\bar{F}(\bar{x}_0) = \bar{0}$ .

$_0$ is a STABLE FIXED POINT if for any neighbourhood, N, of $\bar{x}_0$, there is another neighbourhouc
' such that any trajectory starting in N' stays forever in N.



## ASYMPTOTICALLY STABLE FIXED POINTS

$\bar{F}$ has continuous derivatives ⇒ unique solution

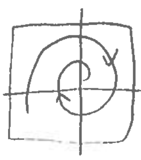
$\bar{x}_0$ is asymptotically stable if there is a neighbourhood
I to $\bar{x}_0$ such that all trajectories starting in N have the property $\bar{x}(t) \xrightarrow[t\to\infty]{} \bar{x}_0$

## NEUTRALLY STABLE FIXED POINT

Is one that is stable but not asymptotically stable.

x) $\begin{cases} x_1' = \overbrace{-x_1 + x_2}^{\text{Linearization}} - x_1^3 \\ x_2' = -x_1 - x_2 + x_2^3 \end{cases}$ origin is a fixed point. $\bar{x}' = A\bar{x}$ , $A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$

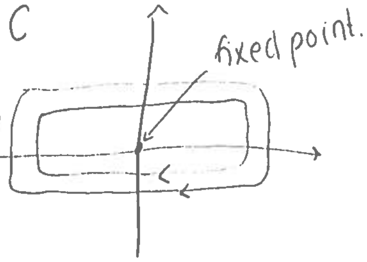
$\lambda^2 + 2\lambda + 2 = 0$  $\lambda_{1,2} = -1 \pm i$



origin of
=> non-linear system is asymptotically stable

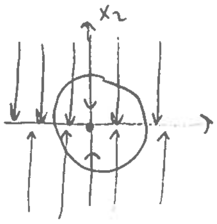
$\begin{cases} x_1' = x_2 \\ x_2' = -x_1^3 \end{cases} \xrightarrow{\text{lineari-zation}}$ $\bar{x}' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \bar{x}$ degenerate!

$\frac{dx_2}{dx_1} = -\frac{x_1^3}{x_2}$  $\int x_2 dx_2 = -\int x_1^3 dx_1$  $\frac{1}{2}x_2^2 = -\frac{x_1^4}{4} + \text{Const}$  $x_1^4 + 2x_2^2 = C$



not asymptotically stable!
STABLE though.

' $x_1' = 0$
$x_2' = -x_2$



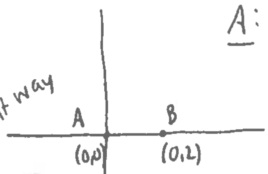
origin neutrally stable
only two trajectories tend to there.

<u>picture in the book</u>
3 fixed points.

there are more than one system having this local behaviour.

ex) $\begin{cases} x_1' = 2x_1 - x_1^2 \\ x_2' = -x_2 + x_1 x_2 \end{cases}$ Fixed points: $x_1(2-x_1) = 0 \Rightarrow x_1 = 0$ or $x_1 = 2$

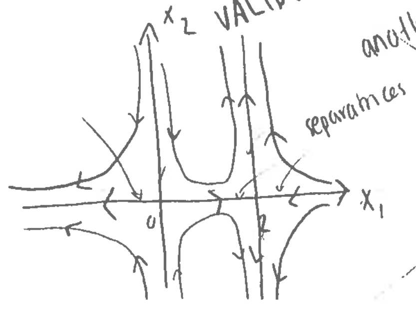
$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases}$ $\begin{cases} x_1 = 2 \\ x_2 = 0 \end{cases}$

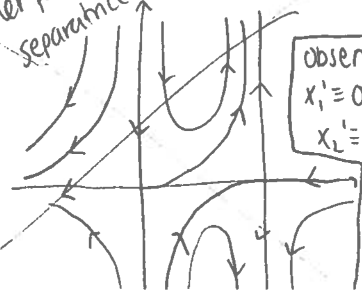


saddle point

A: $\bar{x}' = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \bar{x}$ - linearization in $A = (0,0)$

axises are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.





observe that
$x_1' \equiv 0$ for $x_1 = 0$
$x_2' \equiv 0$, $x_2 = 0$
tells that the first one is the correct

B: $\begin{bmatrix} 2-2x_1 & 0 \\ x_2 & -1+x_1 \end{bmatrix}\Bigg|_{(2,0)} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix}\Bigg|_{(2,0)} =$

$= \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

saddle point.

# 14-11 Stochastic modeling after Gillespie

$_i$, $i=1,\dots,N$ (particles)

$\longrightarrow$ reaction products

$\rightarrow$ — " — (a)

$_i + S_h \longrightarrow$ — " — (b)

$S_i \longrightarrow$ — " — (c)

$S_i + S_h \longrightarrow$ — " — (d)

$_i + S_h + S_j \rightarrow$ — " — (e)

$\{X_i\}_{i=1}^{N}$ current number of particles $S_i$.

$R_H$ reactions $H=1,\dots,M$

Main hypothesis:

$C_H$ reaction parameter

$\underset{(\Delta t)}{\boxed{C_H \cdot \delta t}} + \underset{\uparrow (\Delta t)}{O(\delta t)}$, $\underset{(\Delta t\to 0)}{\delta t \to 0}$ average probability that the reaction $R_H$ will occur during time $\delta t \to 0$.

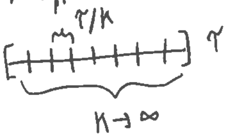
a fcn that divided by $\delta t$ still goes to zero when $\delta t \to 0$.

Main variable in the model

$P(\tau,H)d\tau$ is a probability that after a "waiting time" $\tau$ without any reaction the reaction $R_H$ will occur in time interval $(t+\tau, t+\tau+d\tau)$

$P(\tau,H) = P_0(\tau)\cdot P_H$

probability of waiting time $\tau$.



probability that reaction $R_H$ will occur during time $(\tau/k)\to 0$ is $C_H\cdot \tau/k$ for one set of particles necessary for the reaction $R_H$.

$P_H = C_H \cdot h_H$ number of distinct combinations of $S_i$ in reaction $R_H$.

a) $h = X_i$

b) $h = (X_i)\cdot(X_n)$

c) $h = \frac{X_i(X_i-1)}{2}$

d) $h = \frac{X_i(X_i-1)X_n}{2}$

e) $X_i \cdot X_h \cdot X_j$

$h_H\cdot C_H\cdot\left(\frac{\tau}{k}\right)$ - probability of reaction $R_H$ during time $\frac{\tau}{k}$ for particular numbers $\{X_i\}$ at the moment.

$1 - h_H C_H\left(\frac{\tau}{k}\right)$ probability that reaction $R_H$ will not occur during the time interval $\tau/k$. All reactions are independent $\Rightarrow$ $\prod_{H=1}^{M}\left[1-h_H C_H \frac{\tau}{k}\right] = $ is the probability that no one reaction occurs. $\underbrace{+O\left(\frac{\tau}{k}\right)}$

$$=\left[1-\sum_{H=1}^{M} h_H C_H\left(\frac{\tau}{k}\right) + O\left(\frac{\tau}{k}\right)\right]$$

have put the rest in the ordo term.

$$P_0(\tau) = \left(1-\sum_{H=1}^{M} h_H C_H\left(\frac{\tau}{k}\right)\right)^k = \left(1-\frac{(\sum h_H C_H)\tau}{k}\right)^k \longrightarrow \exp\left\{-\left(\sum_{H=1}^{M} h_H C_H\right)\tau\right\}$$

$$P(\tau,H)d\tau = h_H C_H \exp\left\{-\tau\left(\sum_{H=1}^{M} h_H C_H\right)\right\}d\tau \quad , 0\le\tau<\infty,\ H=1,\dots,M.$$

$$P(\tau,H) = h_H C_H \exp\left\{-\tau\sum_{H=1}^{M} h_H C_H\right\} \quad ,\tau\in[0,\infty[ \quad H=1,\dots,M$$

$$\sum_{H=1}^{M}\int_0^\infty P(\tau,H)\,d\tau = \int_0^\infty\left(\sum_{H=1}^{M} h_H C_H\right)\exp\left\{-\tau\sum_{H=1}^{M} h_H C_H\right\}d\tau = 1$$
$$= \int_0^\infty e^{-\lambda}d\lambda = 1.$$

## Simulation

. Initialisation. Introduce arrays for $\{C_H\}_{H=1}^{M}$ and write in values.

— " — $\{X_i\}_{i=1}^{N}$ — " —

— " — $\{h_H\}_{H=1}^{M}$ and calculate $h_H(X_i, X_h, \dots)$ those $S_i, S_h$ reacting in $R_H$.

2. Generate two random numbers $r_1, r_2$ uniformly distributed over $[0,1]$, $r_1 = \text{rand}()$, $r_2 = \text{rand}()$. used to calculate $\tau$ and $H$, distributed according to $P(\tau,H)$

Change $\{X_i\}_{i=1}^{N}$ according to the reaction $R_H$

$_1 + 2S_2 \longrightarrow S_4 + S_2$ , reaction products.

$$\boxed{\begin{matrix} X_1 \to X_1 - 1 \\ X_2 \to X_2 - 1 \\ X_4 \to X_4 + 1 \end{matrix}}$$

$\underline{t \to t + \tau}$ → recalculate $h_\mu$.

. (output)

$\{t_n\}_{n=1}^{p}$ observation times.

. if we passed next observation time. Then we write out the values $\{X_i\}_{i=1}^{N}$.

heck if we passed next observation time is reached => terminate calculation

heck if the last observation time is reached

therwise go to step 2.

nversion generating method

et $P(x)$ be probability density function for random variable $x$.

$\int P(x)dx$ - probability that $x \in [a,b]$. DEFINITION.

$F(x) = \int^{x} P(\tau)d\tau$ probability distribution function. $F(-\infty) = 0$, $F(+\infty) = 1$

Let $r$ be random uniformly distributed on $[0,1]$. Take $x$ s.t. $\underline{F(x) = r}$. or $x = F^{-1}(r)$, which is

ossible because $F(x)$ is monotone (such functions have inverses).

robability that $x \in ]x', x'+dx'[$ is the same as $r \in ]F(x'), F(x'+dx')[$

$F(x'+dx') - F(x') = \left(\frac{d}{dx'}F(x')\right)dx' = P(x')dx'$

Newton-Leibnitz formula.

we need to simulate distribution $P_0(\tau) = a\exp(-a\tau)$ , $a = \sum h_\mu c_\mu$

$F(t) = \int_0^t a\exp(-a\tau)d\tau = 1 - \exp\{-at\} = r$ → uniformly distributed on $[0,1]$

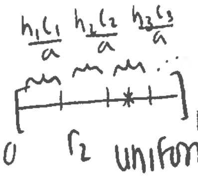
$\exp(-at) = 1 - r \Rightarrow \ln(1-r) = -at \Rightarrow t = \frac{1}{a}\ln\left(\frac{1}{1-r}\right)$ $\tilde{\tau} = \frac{1}{a}\ln\left(\frac{1}{r_1}\right)$

waiting time.

choosing random $\mu$

$P(\tau, \mu) = \left(\frac{h_\mu c_\mu}{a}\right)\left[a\exp\{-a\tau\}\right]$ $\left(\sum \frac{h_\mu c_\mu}{a}\right) = 1$

something is happening. $a = \sum h_\mu c_\mu$



prob. of entering one of the intervals is exactly its length.

uniform on $[0,1]$

in which interval does it end up?

$X \underset{c_2}{\overset{c_1}{\rightleftarrows}} Y$

$W + X \underset{c_6}{\overset{c_5}{\rightleftarrows}} 2X$

$2X \underset{c_4}{\overset{c_3}{\rightleftarrows}} Z$

write down de for nbr of particles.

$$\frac{dX}{dt} = -\underset{c_1 h_1}{c_1X} + \underset{c_2h_2}{c_2Y} - c_3\underbrace{\frac{X(X-1)}{2}}_{h_3}2 + c_4Z + c_5XW - c_6\frac{X(X-1)}{2}$$

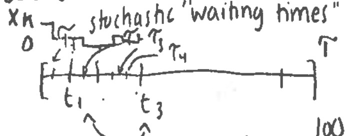
two x are eliminated in the reaction $R_3$

$\frac{dY}{dt} =$

$\frac{dZ}{dt} =$

$\frac{dW}{dt} =$

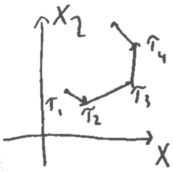
714-11 Comments on Gillespies method



observations. look when passing the $t_i$ what value of $x_n$ we have.

To get a more accurate value, averaged information, we need to run the same initial data several times. $\{\bar{X}^{(s)}(t)\}_{s=1}^{N}$

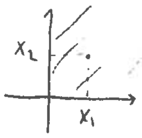
$$\bar{X}(t) = \frac{1}{N}\sum_{s=1}^{N}\bar{X}^{(s)}(t)$$



$x_1, x_2$ not very large

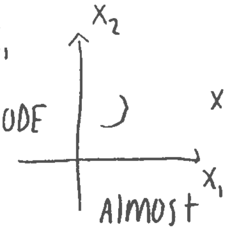
Home assignment 2

. choose an ODE in the plane with polynomial of order 2-3 with a stable fixed point $(x_1>0, x_2>0)$



. Solve the ODE numerically and draw a couple of typical trajectories close to the fixed point.

Almost ODE



$x_1 >> 1$

Almost continuous trajectory

. Model the same system by Gillespie method. Draw random trajectories starting from the same points as ODE and compare them.

4. Scale the time variable $t_1 = 100t$ and repeat the same as in 3.

smaller nbr of particles in one case and larger in the other.

$\frac{dx_1}{dt} = C_1 x_1 x_2 +$ $= \frac{C_1(10x_1)(10x_2)}{100}$ $\frac{dx_1}{dt/100} = (10x_1)(10x_2)$

Random variable, $\xi$ (might be discrete or continuous)

$\xi_n \sim p_n$ - probability to observe $\xi_n$ value, Discrete case. $\{\xi_n\}_{n=1}^{M}$

$M(\xi)$ expectation of $\xi$ $\sum_{n=1}^{M}\xi^{(n)}p_n$ $\sum_{n=1}^{M}p_n = 1$

total probability is one.

$\xi \in D$

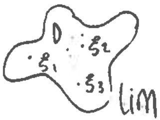
Probability measure on D with density $p(x)$ such that $\int p(\bar{x})d\bar{x} = 1$

fcn of $\xi$

$M(\xi) = \int_D x\, p(x)dx$ in particular $M(g(\xi)) = \int_D g(x)p(x)dx$

we like to compute integrals like $\int_D g(x)p(x)dx \longrightarrow = M(g(\xi)) \approx \frac{1}{N}\sum_{i=1}^{N} g(\xi_i)$ random realization of $\xi$. similarly distributed.

$P(|M(g(\xi)) - \frac{1}{N}\sum_{i=1}^{N} g(\xi_i)| > \varepsilon) \to 0,\ N\to\infty.$ (relatively weak convergence)

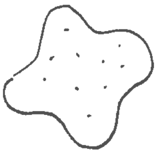


Central limit theorem

$$\lim_{N\to\infty} P\left\{\left(\frac{1}{N}\sum_{i=1}^{N}\xi_i - M(\xi)\right) < x\sqrt{D(\xi)/N}\right\} = \Phi(x)$$ choose probability => estimate for the error.

observations. variance of $\xi$. $\Phi(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-t^2}dt.$

$D(\xi) \approx \frac{1}{N}(\sum \xi_i^2 - (\sum \xi_i)^2)$



$\int_D g(x)p(x)dx \approx \frac{1}{N}\sum_{i=1}^{N} g(\xi_i)$

Integral equations of second type

$Z(p) = \int_G k(p,p')Z(p')dp' + f(p)$

could be the probability of jumping from p' to p.

$\Leftrightarrow$ $Z = \mathbb{K}Z + f$

$Z = \sum_{l=0}^{\infty}\mathbb{K}^l f$ - Neumann series. converges if $\int_G\int_G |k(p,p')|dpdp' < 1$

$(\mathbb{K}^l f)(p_0) = \int \cdots \int k(p_0,p_1)k(p_1,p_2) \quad k(p_{l-1}p_l) f(p_l)dp_1 \quad dp_l \quad \boxed{G^l}$

$\int (K^l t)(p_0) \Psi(p_0)\, dp_0 \longrightarrow$ a number $\quad (z, \Psi) = \int_G z(p_0) \Psi(p_0)\, dp_0 = \int_G \left( \sum_{l=0}^{\infty} K^l t \right) \Psi(p_0)\, dp_0$

we need a random point in $\underbrace{G \times \dots \times G}_{l+1}$, $(\underset{G}{\underset{\cap}{Q_0}} \to Q_1 \to \dots \to Q_l)$

$p(P, P')$ probability density to jump from $P$ to $P'$. we choose also a probability density $p(P)$ for initial point $p_0$.

$Q_0$ - random point with density $p(P)$

$Q_1$ is a random point with density $p(Q_0, P')$

ntroduce $W_j = \dfrac{K(Q_0, Q_1) K(Q_1, Q_2) \dots K(Q_{j-1}, Q_j)}{p(Q_0, Q_1) p(Q_1, Q_2) \dots p(Q_{j-1}, Q_j)}$, $W_j = W_{j-1} \dfrac{K(Q_{j-1}, Q_j)}{p(Q_{j-1}, Q_j)}$

$\Theta(\psi) = \dfrac{\psi(Q_0)}{p(Q_0)} W_l \varphi(\Theta_l) \qquad M(\Theta_l(\psi)) = \int K^l \varphi(P) \psi(P)\, dp.$

$$\int_G \int_G \dots \int_G \frac{K(P_0, P_1) K(P_1, P_2) \dots K(P_{l-1}, P_l)}{p(P_0, P_1) p(P_1, P_2) \dots p(P_{l-1}, P_l)} \cdot p(P_0, P_1) p(P_1, P_2) \dots p(P_{l-1}, P_l) f(p_l) \Psi(P_0)\, dp_1 \dots dp_l$$

$Q_l$

same idea as $\int_D f(x)\, dx = \int_D \left( \dfrac{f(x)}{p(x)} \right) p(x)\, dx = M\left( \dfrac{f(\xi)}{p(\xi)} \right)$ which can be estimated by LLN.

can be choosen almost arbitrary.

11/4-11

$N + X \underset{C_6}{\overset{C_5}{\rightleftarrows}} 2X$ Comments on Home Assignment N2.

$X \xrightarrow{C_1} *$

write ODE for the number of particles.

$$\frac{dW}{dt} = -C_5 W X + C_6 \left(\frac{1}{2} X(X-1)\right)$$

$x^2$ for large $x$.

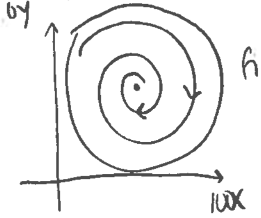
$$\frac{dX}{dt} = C_5 W \cdot X - C_6 \frac{1}{2} X(X-1) - C_1 X$$

Write a code in Matlab using ginput() function for choosing initial point and solving ODE using ODE45( ). Plot solutions. plot(x,y) [t,x,y] = ode45( ) in the first quadrant. (x,y)-positions at time t

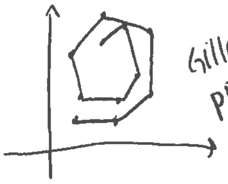
choose a system with a stable fix point.

2. solve the same system using Gillespie's method.
Make graphical output in the way similar to one you had with ODE: plot(x,y) → vector of position at random times.

The phase portrait from ODE will be "interesting" at some area like $[0,10] \times [0,10]$



fixed point



Gillespie picture

scaling to see effect of large number of particles

$$\frac{\partial x}{\partial t} = 2x + 3y \quad \Big| \; 100$$
$$\frac{\partial y}{\partial t} = -x + y \quad \Big| \; 100$$

like to introduce $X = 100x$, $Y = 100y$

Linear system

$$\frac{dX}{dt} = 2X + 3Y$$
$$\frac{dY}{dt} = X + Y$$

scaled system is the same

Scaling for non-linear system

$$\frac{dW}{dt} = -C_5 W \cdot X + C_6 \frac{1}{2} X^2 \quad \Big| \; 100$$
$$\frac{dX}{dt} = C_5 W X - C_6 \cdot \frac{1}{2} X^2 - C_1 X \quad \Big| \; 100$$

$\bar{W} = 100w$
$\bar{X} = 100x$

$$\frac{d\bar{W}}{dt} = -\frac{C_5 \bar{W} \bar{X}}{100} + \frac{C_6}{2 \cdot 100} \bar{X}^2$$
$$\frac{d\bar{X}}{dt} = \frac{C_5}{100} \bar{W} \bar{X} - \frac{C_6}{2 \cdot 100} \bar{X}^2 - C_1 \bar{X}$$

$w = \frac{\bar{W}}{100}$
$x = \frac{\bar{X}}{100}$

$$\int_0^\infty f(x) e^{-hx} dx = \int_0^\infty \frac{1}{K} f(x) \boxed{h e^{-hx}} dx = M\left(\frac{1}{h} f(\xi)\right) \approx \text{for } \xi \text{ with probability density } p(x).$$

prob. density on $[0,\infty[$ $p(x) = he^{-hx}$, $\int_0^\infty p(x) dx = 1$

$$\approx \frac{1}{N} \sum_{i=1}^{N} \frac{1}{K} f(\xi_i), \quad \xi_i = -\frac{1}{K} \ln \gamma_i \text{ for } \gamma \in [0,1], \; \gamma \text{ uniformly distributed on } [0,1].$$

Monte Carlo methods converges as $\frac{1}{\sqrt{N}}$ only for $N \to \infty$, but in ANY dimension.

## Importance sampling

$\int_0^{\infty} g(x)dx$

ie know that $g \sim e^{-x^2}$ for large x. $(x \to \infty)$ choose $p(x) = const \cdot e^{-x^2}$ so that $\int_0^{\infty} e^{-x^2} const\, dx = 1$.

we imitate the situation above. $\int_0^{\infty} g(x)dx = \int_0^{\infty} \left(\frac{g(x)}{p(x)}\right) p(x) dx = M\left(\frac{g(\xi)}{p(\xi)}\right) \approx \frac{1}{N}\sum_{i=1}^{N} \frac{g(\xi_i)}{p(\xi_i)}$

for $\xi$ with distribution $p(x)$ over $[0, \infty[$

---

$z(P) = \int_G K(P, P') z(P') dp' + f(P)$

$z = \mathbb{K}z + f$ in operator form. Iterations for solving the eq.

$z^{(0)} = \varphi(p)$ initial approximation. $z^{(i+1)} = \mathbb{K}z^{(i)} + f \Rightarrow z = \sum_{i=1}^{\infty} \mathbb{K}^i \cdot f$

i:th iteration $z^{(i)} = f + \mathbb{K}f + \dots + \mathbb{K}^{i-1}f + \mathbb{K}^i\varphi$.

We compute functionals of solutions. $\int_G \Psi(P) z(P) dp$

$\int K^i \varphi(p)\Psi(p)dp = \int_G dP_0 \int_G dP_1 \int_G dP_2 \cdot \int_G dP_i\ K(P_0, P_1) K(P_1, P_2) \dots K(P_{i-1}, P_i)\varphi(P_1)\Psi(P_0)$

$\bar{\mathbb{P}} = (P_0, P_1, \dots, P_i)$

$\int_G \overset{\leftarrow \text{operator}}{K^i} \varphi(P)\Psi(P)dp \approx \frac{1}{N}\sum \{\mathbb{K}(\mathbb{P}_i)\}$

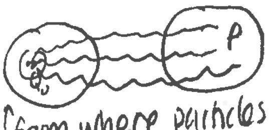
choose the density for $\mathbb{P}_i$ as a product of densities $p(\overset{\text{parameter, known, from where we jump to next point}}{P}, P') = p(P \to P')$ for given P generate a random P'.

$P_j = p(\underbrace{Q_0}_{\text{fixed before}}, \underbrace{Q_1) p(Q_1, Q_2) \dots p(Q_{j-1}, Q_j)}_{\text{modelled}}$

$w_{j-1} = w_j \frac{K(Q_{j-1}, Q_j)}{P(Q_{j-1}, Q_j}$ $\int \mathbb{K}^i \varphi(P)\Psi(P)dp \approx \frac{1}{N}\sum_{i=1}^{N} \Theta_i(\Psi)$ $\Theta_i(\Psi) = \frac{\Psi(Q_0)}{p(P_0)} w_i \varphi(Q_1)$, $Q_0$ has density $p(P_0$

$\int_G z(P)\Psi(P)dp \approx \frac{1}{N}\sum_{s=1}^{N} \zeta[\Psi]_s$ $\xi = \left[\underbrace{\frac{\Psi(Q_0)}{P(Q_0)}}_{\text{one time}}\right] \sum_{i=0}^{\infty} w_i f(Q_i)$ ← right hand side is computed many times. (initial data).

start in a point and look back from where the system might have come to there



from where particles might come.

## Conjugate equation and "forward" Monte Carlo

$u(P) = \int_G K^*(P, P') u(P') dp' + \Psi(P)$ $K^*(P, P') = K = (P', P)$ $(\mathbb{K}^* u, z) = (u, \mathbb{K}z)$

$\int_G uz\,dp = \int_G u[Kz + f]dp = \int_G (zK^*u + fu)\,dp$

$\int_G uz\,dp = \int_G z[K^*u + \Psi]dp = \int_G (zK^*u + \Psi z)\,dp$

$\Rightarrow \int_G z\Psi\,dp = \int_G fu\,dp.$

18/4-11 First integrals

DEF · $\bar{x}' = \bar{X}(\bar{x})$; Function $f(\bar{x})$ is a first integral to the ODE above in a domain D if it is constant on all trajectories in D.

- If the ODE has a first integral in the whole $\mathbb{R}^n$ it is called conservative
- In many examples of conservative systems first integral has meaning of enegy.

Idea how to find first integrals

$$\frac{dx_1}{dt} = x_1' = \bar{X}_1(x_1, x_2)$$
$$\frac{dx_2}{dt} = x_2' = \bar{X}_2(x_1, x_2)$$
$$\Rightarrow \frac{dx_2}{dx_1} = \frac{\bar{X}_2(x_1, x_2)}{\bar{X}_1(x_1, x_2)} \Rightarrow ?\ \bar{X}_1(x_1,x_2)dx_2 = \bar{X}_2(x_1,x_2)dx_1$$

level curves. $f(x_1, x_2) = C$


must consi of pieces of traject-ones

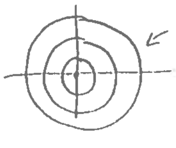
examples

$\dot{x}_1 = -x_2 ;\ \dot{x}_2 = x_1 \Rightarrow$ conservative

$\dot{x}_1 = x_1 ;\ \dot{x}_2 = x_2 \Rightarrow$ non-conservative

$$\frac{dx_1}{dt} = -x_2,\ \frac{dx_2}{dt} = x_1 \Rightarrow \frac{dx_2}{dx_1} = \frac{-x_1}{x_2} \Rightarrow \int -x_2 dx_2 = \int x_1 dx_1$$

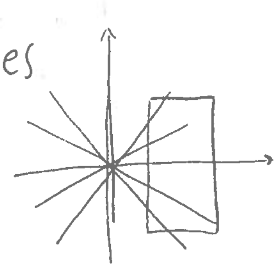
$$-\frac{x_2^2}{2} = \frac{x_1^2}{2} + \text{const} \Rightarrow f(x_1,x_2) = x_1^2 + x_2^2 = \text{constant (on trajectories)}$$


level sets of f are circles and they are trajectoric (since the origin is the only fixed point)

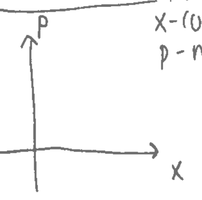
$$\frac{dx_1}{dt} = x_1,\ \frac{dx_2}{dt} = x_2 \Rightarrow \frac{dx_2}{dx_1} = \frac{x_2}{x_1}$$

$\frac{x_2}{x_1}$ = const on trajectories

not continuous in $\mathbb{R}^2$



Hamiltonian systems



x-coordinates
p-momentum

$H(x,p)$ Hamiltonian

$$x' = \frac{\partial H(x,p)}{\partial p} \quad p' = -\frac{\partial H(x,p)}{\partial x}$$

It is a first integral and has the meaning of energy.

$$\frac{d}{dt} f(\underset{\text{trajectory}}{\bar{x}(t)}) = \nabla f \cdot \bar{X}(\bar{x}(t)) = 0$$ if f is a first integral.

$$\frac{d}{dt} H(x,p) = \frac{\partial H}{\partial x}\underset{\uparrow \frac{\partial H}{\partial p}}{\frac{\partial x}{\partial t}} + \frac{\partial H}{\partial p}\underset{\uparrow -\frac{\partial H}{\partial x}}{\frac{\partial p}{\partial t}} = \frac{\partial H}{\partial x}\cdot\frac{\partial H}{\partial p} - \frac{\partial H}{\partial x}\frac{\partial H}{\partial p} = 0$$

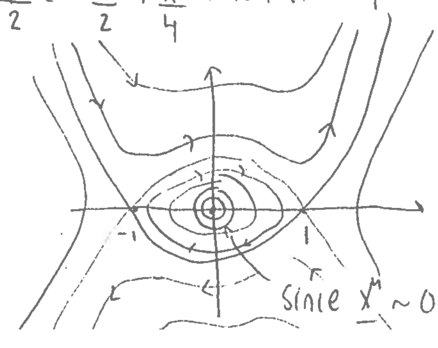
Example Find first integral

$x' = p ;\ p' = -x + x^3$ (linearization doesn't help) center!

$$\frac{\partial p}{\partial x} = \frac{-x + x^3}{p} \Rightarrow \int p\,dp = \int (-x + x^3)dx + \text{const}$$

$$\frac{p^2}{2} = -\frac{x^2}{2} + \frac{x^4}{4} + \text{const.} \quad p^2 + x^2 - \frac{x^4}{2} = \text{const}$$ defined in the whole plane (also a Hamiltonian!)

can recognize centers using first integrals



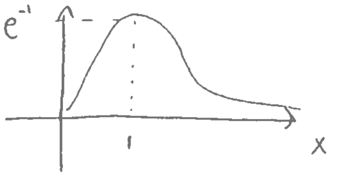
since $x^4 \sim O(x^2)$ center in the

xample show that the system has a center

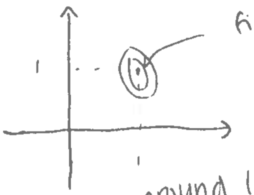
$x_1' = x_1 - x_1x_2$
$x_2' = -x_2 + x_1x_2$

$$\frac{dx_2}{dx_1} = \frac{-x_2 + x_1x_2}{x_1 - x_1x_2} = \frac{x_2}{x_1}\frac{(x_1-1)}{(1-x_2)} \Rightarrow \int \frac{(1-x_2)}{x_2}dx_2 = \int \frac{(x_1-1)}{x_1}dx_1$$

$$\Rightarrow \ln x_2 - x_2 = x_1 - \ln x_1 + \text{const} \qquad x_2e^{-x_2} = \frac{1}{x_1}e^{x_1} \cdot \text{const} \qquad g(x) = xe^{-x}$$



$g(x_1)g(x_2) = \text{const}.$



no other stationary points $\Leftarrow$ around (1,1) has to be center

$f(x_1, x_2)$ has maximum in (1,1)

$\nabla f(1,1) = 0$

maximum $\Rightarrow$ $\nabla = 0$.

$$f(1+\Delta x_1, 1+\Delta x_2) = e^{-2} + \cancel{\frac{\partial f}{\partial x_1}(1,1)\Delta x_1 + \frac{\partial f}{\partial x_2}(1,1)\Delta x_2} +$$

$$0 < \left[ + \frac{\partial^2 f}{\partial x_1^2}(1,1)(\Delta x_1)^2 + 2\frac{\partial^2 f(1,1)}{\partial x_1 \partial x_2}\Delta x_1 \Delta x_2 + \frac{\partial^2 f}{\partial x_2^2}(\Delta x_2)^2 \right]$$

$$+ o(\Delta x_1^2, \Delta x_2^2$$

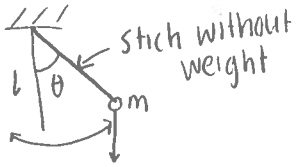
$$\text{const} = a\Delta y_1^2 + b\Delta y_2^2$$

## Topics for exam

5 problems covering main chapters in the course.

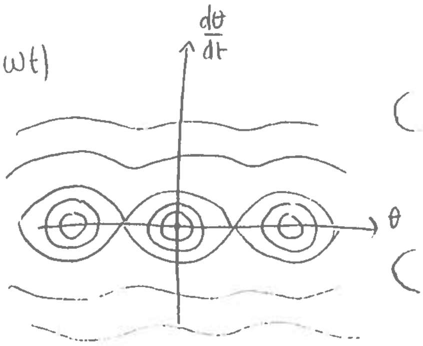
. Linear systems. Types of phase portraits
Evolution operator. Affine systems.
.. Stability of fixed point for non-linear ODE.
tability by linearization. Stability by Lyapuvs functions.
. First integrals
. Periodic solutions. Poincaré - Bendixson theory.
. Hopf bifurcation (formula for stability index NOT required)
Gillespie method for chemical reactions

## athematical pendulum



$$l\frac{d^2\theta}{dt^2} = -g\sin\theta + (\varepsilon\cos\omega t)$$

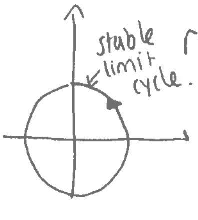
$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)}$$



## Periodic solutions Limit cycles

x) $x_1' = -x_2 + x_1[1 - (x_1^2 + x_2^2)^{1/2}]$
$x_2' = x_1 + x_2[1 - (x_1^2 + x_2^2)^{1/2}]$

$x_1 = r\cos\theta$ $\theta' = 1$
$x_2 = r\sin\theta$ $r' = r(1-r)$ (multiply first by $x_1$ and second by $x_2$ and add)



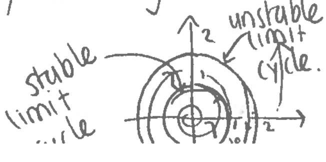
$r = 1$, $\theta = t + \theta_0$ periodic solution in $(x_1, x_2)$

$r > 1 \Rightarrow r' < 0$ trajectory tends to $r = 1$

$r < 1 \Rightarrow r' > 0$

EF: Limit cycle is a periodic solution that is isolated. There is a tabular domain around i
ithout other periodic solutions

f any trajectory starting close to a limit cycle tends to it with $t \to \infty$, the limit cycle is
alled stable.



r increases
$r > 2$, $r' > 0$
$1 < r < 2$ $r' < 0$

$$\begin{cases} r' = r(r-1)(r-2) \\ \theta' = 1 \end{cases}$$

$r < 0$ r increases

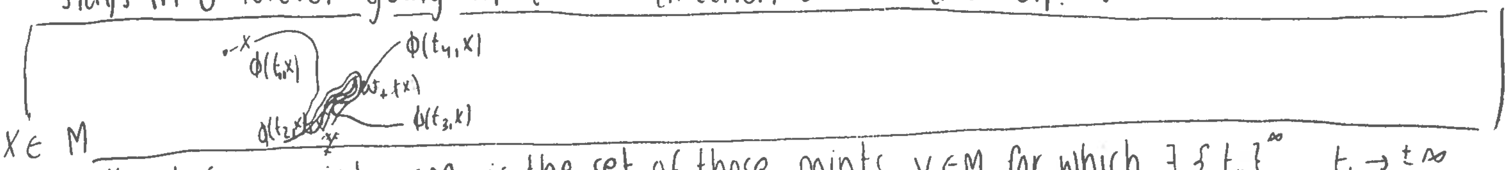
15-11 Theorem 3.9.1 and Corollary after it on page 110 in Arrowsmith Place book are WRONG!
We follow book by Taschle pages 149-151 and 187-191.

oincaré - Bendixson Theorem and Criterion

efinitions and formulations are included in examination and also the proof of Bendixsons riterion, but NOT proof of the Poincaré - Bendixsons Theorem.

rbits-Trajectories $\bar{x}' = f(\bar{x})$; $f \in C^1(M)$, where M open set.

EF: A set $U \subset M$ is $\sigma$-invariant for ($\sigma = +$ or $-$) if any orbit $\gamma_\sigma(x) \subset U$ $\forall x \in U$ starting in U stays in U forever going in $t \to +\infty$ direction or $-\infty$ direction. $t \to \sigma\infty$



$\omega_\pm$ limit set for a point $x \in M$ is the set of those points $y \in M$ for which $\exists \{t_n\}_{n=1}^{\infty}$ $t_n \xrightarrow[n\to]{} \pm\infty$ such that $\varphi_{t_n}(x) \to y$; $\phi(t_n, x) \to y$. (large set which attracts the whole trajectory)

$\boxed{\mathbb{R}^2 \text{ ONLY!}}$ P.B. Theorem

Let M be an open set in $\mathbb{R}^2$, $\bar{f} \in C^1(M)$, $x \in M$. Suppose that $\omega_\sigma(x) \neq \phi$ not empty, compact; connected and contains finitely many fixed points Then one of the following cases hold.

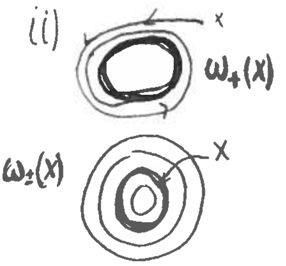
) $\omega_\sigma(x)$ is a fixed point

) $\omega_\sigma(x)$ is a periodic orbit

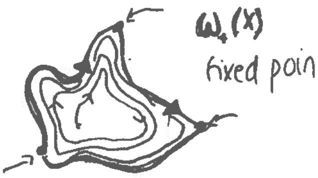
i) $\omega_\sigma(x)$ consists of finitely many fixed points $\{x_i\}$ and unique non-closed orbits $\gamma(y)$ such tha $\omega_\pm(y) \in \{x_i\}$

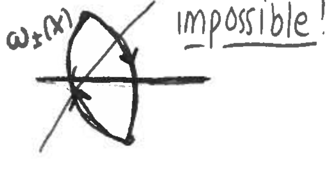
)

ii)



iii)





missed in Arrow-smith Place's book.

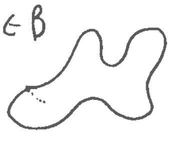
Lemma 6.5 (Taschle)

The set $\omega_\pm(x)$ is a a) closed b) invariant set

$\underbrace{\text{invariant set}}_{}$ $\omega_\pm(x)$ consists of orbits (trajectories)

Closed set B means that for any $\{x_n\}_{n=1}^{\infty} \in B$ $\lim_{n\to\infty} x_n \in B$

$y \in \omega_\pm(x)$



roof: Take $y \in \overline{\omega_\pm(x)}$ ($\omega_\pm(x)$ and its boundary)

ahe $\{y_n\} \in \omega_\pm(x)$ such that $|y_n - y| < (2n)^{-1}$

ahe $t_n \to \pm\infty$ s.t. $|\overset{\text{point on trajectory.}}{\phi(t_n, x)} - y_n| < (2n)^{-1}$ (can be done because of the def of the limit set)

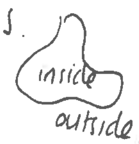
riangle inequality

$\Rightarrow$ $|\phi(t_n, x) - y| < \frac{1}{n} \xrightarrow[n\to\infty]{} 0$

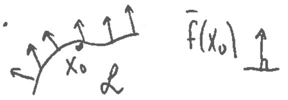
) $y \in \omega_\pm(x) \overset{\text{def}}{\Rightarrow} \exists \{t_n\}_{n=1}^{\infty} : \phi(t_n, x) \xrightarrow[n\to\infty]{} y$ ($\phi(\ldots,\ldots)$ is continuous with respect to starting point)

$\phi(t_n + t, x) = \phi(t, \underbrace{\phi(t_n, x)}_{y,\, n\to\infty}) \xrightarrow[n\to\infty]{} \phi(t, y)$

he main difference between $\mathbb{R}^2$ and $\mathbb{R}^n$, $n>2$, is demonstrated by Jordan's curve Theorem.
A closed continuous curve without intersections divide $\mathbb{R}^2$ into two disjoint open sets.



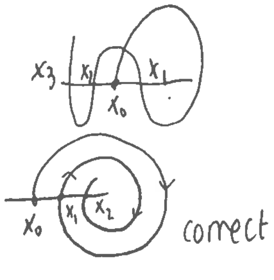
F: Transversal arc to a point $x_0$ (that is not a fixed point) is a (short) curve $\mathcal{L}$ ncluding $x_0$ such that for all $x \in \mathcal{L}$, $\bar{f}(x)$ point to the same side of $\mathcal{L}$ and are not tangent $\mathcal{L}$.



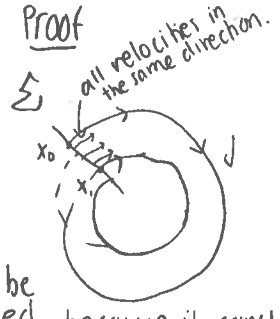
### Lemma 8.1

If $x_0$ is not a fixed point and $\Sigma$ is a transversal arc containing $x_0$, denote by $x_n$ crossing points for the orbit $\phi(t, x_0)$.

$x_n$ must be monotone on the arc



Proof

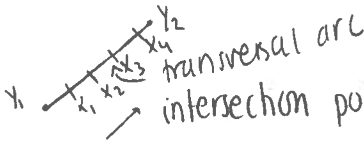


J : part of trajectory between $x_0$ and $x_1$.

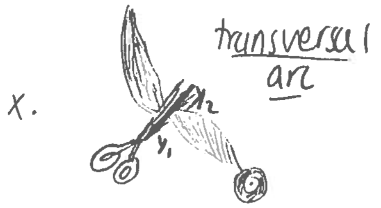
can't be realized, because it can't cross the trajectory.

### Corollary 8.2

Let $\Sigma$ be a transversal, then $\omega_\sigma(x)$ intersects $\Sigma$ only at one point



transversal arc
intersection points as before



Corollary 8.3 $\omega_\sigma(x) \cap \gamma_\sigma(x)$ => x is periodic and $\omega_\sigma(x) = \gamma_\sigma(x)$

orollary to Poincaré-Bendixson Theorem -

f U is a compact invariant set for $\bar{x}' = f(\bar{x})$ in plane and U doesn't include any fixed oints, then for any $x \in U$, $\omega_\pm(x)$ is a closed orbit (or periodic solution).

orollary -

ny closed orbit in plane must contain a fixed point inside.

5/5-2011 corollary (a corrected version)

P.110 in A.P

go to infinitely large positive times, $t\to\infty$

Consider a bounded closed region positively invariant set D for $\bar x' = \bar f(\bar x)$,  $\bar f\in C^1(M)$, $D\subset M$

$\underbrace{\text{bounded closed}}_{\text{compact.}}$

without fixed points. The set D must include at least one closed orbit (periodic solution)

Theorem 3.9.2 in A.P

Let D be a simply connected region in the plane where the system $\bar x' = \bar f(\bar x)$; $\bar f\in C^1(\bar D)$

If div $\bar f$ has a constant sign (+ or -) in D, then there are no periodic solutions (closed orbits) in D

Proof by contradiction

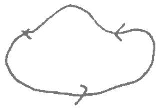
Suppose there is a periodic solution $\{x_1(t), x_2(t)\}$ $x_1(t+T) = x_1(t)$; $x_2(t+T) = x_2(t)$

Use Gauss' Theorem

$$\int_{\partial\Omega}\bar f\cdot\bar n\,dl \neq \int_{\Omega}\operatorname{div}\bar f\,d\bar x$$

$V_0$ if div $\bar f>0$

right hand side in the equation



$\bar f$ is everywhere tangent to the boundary since it is the velocity matrix.

example 3.9.1 Show that the phase portrait of the eq. $x'' - x'(1-3x^2-2(x')^2)+x=0$ has a closed orbit.

solution $x_1' = x_2$

$x_2' = -x_1 + x_2(1-3x_1^2-2x_2^2)$

Find a compact positively invariant set without fixed points

why bounded invariant set?

points.

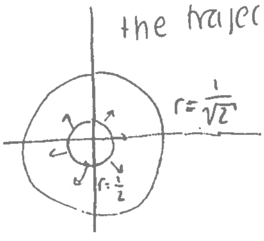
Advise! Find a positively invariant ring $a<r<b$ without fixed points.

write down an equation for r in polar coordinates.

Multiply $x_1'$ by $x_1$, $x_2'$ by $x_2$ and add! $r' = r\sin^2\theta(1-3r^2\cos^2\theta-2r^2\sin^2\theta)$

$\theta' = -1+\frac{1}{2}\sin 2\theta(1-3r^2\cos^2\theta-2r^2\sin^2\theta)$

$r_0<<1$ can choose $r_0$ so small that the trajectory



Take $r=\frac12$ $r' = \frac14\sin^2\theta(1-\frac12\cos^2\theta)$

$\vee_0$

$r'\leq r\sin^2\theta(1-2r^2\cos^2\theta-2r^2\sin^2\theta) = r\sin^2\theta(1-2r^2)\leq 0$ when $r>\frac{1}{\sqrt2}$ $\Rightarrow$

will just lea the circle

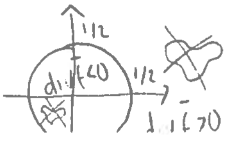
$\frac12<r<\frac{1}{\sqrt2}$ is a positively invariant set without fixed points. Origin is the only fixed point of the system.

Ex 3.9.2 Prove that the system $\begin{cases} x_1' = -x_2+x_1(1-x_1^2-x_2^2) = f_1(\bar x) \\ x_2' = x_1+x_2(1-x_1^2-x_2^2)+k = f_2(\bar x)\end{cases}$ with k constant can have a closed orbit only if a) this orbit encircle the origin or

b) this orbit intersect the circle $x_1^2+x_2^2 = \frac12$

consider div $\bar f = 1-3x_1^2-x_2^2+1-x_1^2-3x_2^2 =$

$= 2-4(x_1^2+x_2^2)$ $\begin{cases} >0 \text{ if } x_1^2+x_2^2<\frac12 \\ <0 \text{ if } x_1^2+x_2^2>\frac12\end{cases}$



exclude these because of constant divergence Two alternatives

7. Prove that there exist a region $R=\{x_1^2+x_2^2\leq r_0^2\}$ such that the system $x_1'=-wx_2+x_1(1-x_1^2-x_2^2)$
$x_2'=wx_1+x_2(1-x_1^2-x_2^2)-F$
w and F constants) enter R. Show that system has a periodic solution
or F=0. (

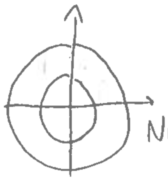
dvise: Try to find an equation for $r'=($ or for something like $(ax_1^2+bx_2^2)'=(\quad)$

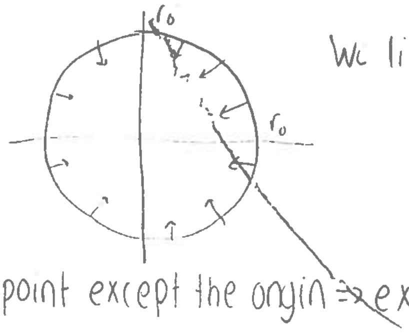
$x_1x_1'+x_2x_2'=\frac{1}{2}(r^2)'=-w\cancel{x_1}x_2+x_1^2(1-x_1^2-x_2^2)+w\cancel{x_1}x_2+x_2^2(1-x_1^2-x_2^2)-Fx_2=r^2(1-r^2)-Fr\sin\theta$

$r'=r(1-r^2)-F\sin\theta$ $\quad|F\sin\theta|<|F|$

We like to find $r_0$: $r_0(1-r_0^2)+|F|<0$

f $r_0>1\Rightarrow r'<0$
f $r_0<1\Rightarrow r'>0$





No fixed point except the origin ⇒ exist closed orbit

18 $\begin{cases}x_1'=1-x_1x_2\\x_2'=x_1\end{cases}$ has no periodic orbits.

(It has no fixed points!) such a system can't have closed orbits. Why?


no more periodic orbits inside.

Positive invariant set, so it must include another periodic orbit but it was the most inner one!

Any periodic orbit must have a fixed point inside, or it will contradic the Bendixson Theorem

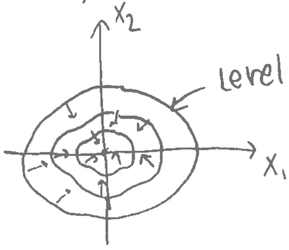
9/5-11 Liapunov functions (stability of stationary points)
(Linearization gives a center)

51 Mayers
1114

$x_1'=-x_1^3$
$x_2'=-x_2^3$

(0,0) fixed point

$V(x_1,x_2)$

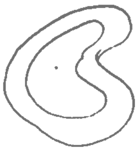


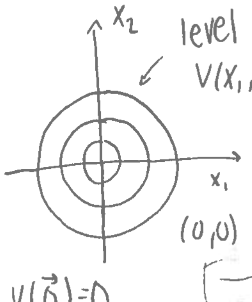
$V(x_1,x_2)=x_1^2+x_2^2$

$$V'(x_1(t),x_2(t))=\nabla V\cdot\begin{bmatrix}-x_1^3\\-x_2^3\end{bmatrix}=\begin{bmatrix}2x_1\\2x_2\end{bmatrix}\cdot\begin{bmatrix}-x_1^3\\-x_2^3\end{bmatrix}=-2(x_1^4+x_2^4)<0$$

$\begin{bmatrix}x_1\\x_2\end{bmatrix}'$

$(x_1,x_2)\neq(0,0)$





level sets of $V(x_1,x_2)$

$(x_1,x_2)\to(0,0)$, $t\to\infty$

(0,0) is asymptotically stable.

$V(\vec{0})=0$

## DEFINITION

A real-valued $V: N\subset\mathbb{R}^2\to\mathbb{R}$, N neighbourhood of (0,0).
is positive (negative) definite if $V(x)>0$ $(V(x)<0)$ $x\neq\vec{0}$
or case with $V(x)\geq0$ $(V(x)\leq0)$ positive (negative)
emi definite.

h 5.4.1. Liapunovs stability theorem.

uppose the system $\bar{x}'=f(\bar{x})$ has a fixed point in the origin and there is a function V in a eighbourhood of the origin such that

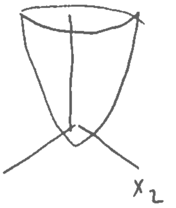
) $V$, $\frac{\partial V}{\partial x_1}$, $\frac{\partial V}{\partial x_2}$ are continuous

) V is positive definite

$V'(x_1,x_2)$ is negative semidefinite ⇒ origin is a stable fixed point if c') $V'(x_1, x_2)$ is negative definite => origin is assympt. stable.

roof

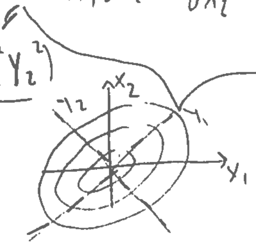
| and b) imply that the level curves of V are continuous closed curves around the origin lose to the origin. why ?? $V(\bar{0})=0$ ; $V>0$ , $\bar{x} \neq \bar{0}$ => origin is a local minimum.



$$V(\bar{x}) = \bar{0} + \underset{0}{\cancel{\nabla \bar{V}}} \cdot \bar{x} + \bar{x}^T \left. \begin{bmatrix} \frac{\partial^2 V}{\partial x_1^2} & \frac{\partial^2 V}{\partial x_1 \partial x_2} \\ \frac{\partial^2 V}{\partial x_1 \partial x_2} & \frac{\partial^2 V}{\partial x_2^2} \end{bmatrix} \right|_{\bar{x}=0} \cdot \bar{x} + O(|x|^3)$$

quadratic form.

in a rotated coordinate system. $(\lambda_1^2 y_1^2 + \lambda_2^2 y_2^2)$



stability

∀ neighbourhood N of $\vec{0}$ we can find a neighbourhood $N_1$ of $\vec{0}$ such that

$\forall x_0 \in N_1$, $\varphi_t(x_0) \subset N$

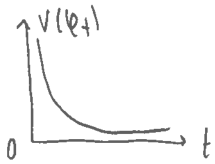
Find a value k for V such that the level set $N_1$ of $k \subset N$. observe that $V'(\bar{x}) \leq 0$.

Two options are possible for an orbit starting in $N_1$ : $\varphi_t(x_0) \to \vec{0}$ or $\varphi_t(x_0) \to$ periodic solution. (impossible if $V'(\bar{x}) < 0$

Another argument:

ook at $V(\varphi_t(x_0))$ ; $V'(\varphi_t(x_0)) < 0$ → $V(\varphi_t(x_0))$ is strictly decreasing. and therefore $\lim_{t \to \infty} V(\varphi_t(x_0)) \to c$

out c can be only zero ($\vec{0}$)



(bounded from below!)

$V(x_1, x_2)$ - Liapunovs functions

$V'(x_1(t), x_2(t)) < 0$ : strong Liapunovs f. (assymptotic stability)

$V'(x_1(t), x_2(t)) \leq 0$ : weak Liapunovs f. (weak stability)

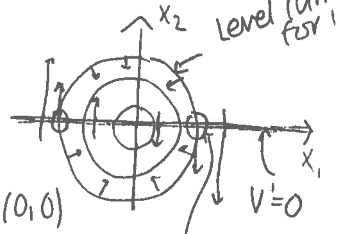
If not in the origin, make a change of variables.

example: $x'' + x'^3 + x = 0$ , $x'=0, x=0$ → fixed point.

$\begin{cases} x_1' = x_2 \\ \boxed{x_2' = -x_1} - x_2^3 \end{cases}$

consider $V(x_1, x_2) = x_1^2 + x_2^2$ - weak Liapunovs fcn.

$V'(x_1(t), x_2(t)) = 2x_1 \cancel{x_2} + 2x_2(\cancel{-x_1} - x_2^3) = -2x_2^4 \leq 0$ $(x_1, x_2) \neq (0,0)$



The origin is stable, but asymptotically stable

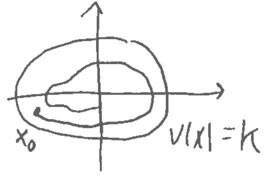
if $V'(x_1(t), x_2(t))$ is not identically 0 on whole trajectories, then even a weak iapunovs function implies asymptotical stability.

the theorem doesn't say anything here (?

heorem 5.4.2 If there exists a weak Liapunov function V for the system $\bar{x}' = f(\bar{x})$ in a neighbourhood of an isolated fixed point at the origin then, providing $V'(\bar{x}) \neq 0$ identically on any rajectory other than the origin itself, the origin is asymptotically stable.

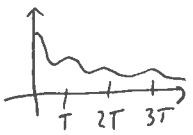
roof: $\exists N_1 \subset N$ , $N_1 = \{x : V(\bar{x}) < k\}$ (where the system is defined.

$\varphi_t(x_0)$ for $x_0 \in N_1$ is contained in $N_1$ because $V'(x) \leq 0$.



) $\varphi_t(x_0) \to 0$ => asymptotically stable.

) $\varphi_t(x_0) \to$ periodic solution $\lim_{t\to\infty} V'(\varphi_t(x_0)) = 0$   contradicts that $V'(x) \neq 0$ identically.

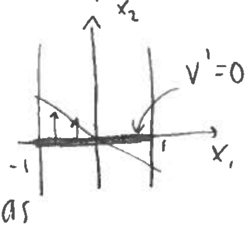


x 5.4.3

$_1' = x_2$   $V(x_1, x_2) = x_1^2 + x_2^2$

$_2' = -x_1 - (1-x_1^2)x_2$   $V'(x_1(t), x_2(t)) = \cancel{2x_1x_2} + 2x_2(\cancel{-x_1} - (1-x_1^2)x_2) = -2x_2^2(1-x_1^2) \le 0$ for $|x_1|<1$

=> origin is stable. $V'=0$ does not include any trajectories → origin is asymptotically stable.



Theorem 5.4.3

uppose $\bar{x}' = f(\bar{x})$ has a fixed point at the origin. If a real-valued function v has roperties that:

) the domain of v contains a disc $N = \{|x| \le r\}, r>0$

) there are points $x_i$ arbitrarily close to the origin where $v(x_i) > 0$.

) v is positive definite.

) $v(0)=0$.   => origin is unstable

12/5-11 included at the end!

16/5-11 Examples on instability.

Show that the system $\begin{cases} x_1' = x_2^2 - x_1^2 \\ x_2' = 2x_1x_2 \end{cases}$ has unstable fixed point in the origin.

1) $V(x_1, x_2): \begin{cases} V(\vec{0}) = 0 \\ V(\bar{x}_i) > 0 \\ \{\bar{x}_i\}_{i=1}^{\infty},\ \bar{x}_i \xrightarrow[i\to\infty]{} 0 \end{cases}$ sequence of points tending to the origin.

2) $V'(x_1(t), x_2(t)) > 0$   $(x_1, x_2) \neq (0,0)$

positive definite   everywhere!

$) < x_2 < \sqrt{3}$ => positive

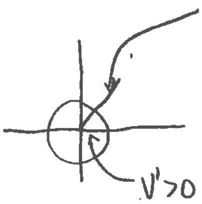
Take $V = 3x_1 \overset{0}{\wedge} x_2^2 - x_1^3 > 0$

$x_1 \ge 1$

$3x_1x_2^2 > x_1^3$   $x_1 > 0$

$x_2 = x_1$

$3x_1^3 > x_1^3$



$V' = (3x_2^2 - 3x_1^2)(x_2^2 - x_1^2) + 6x_1x_2(2x_1x_2) = 3[(x_2^2 - x_1^2)^2 + 4x_1^2x_2^2] = 3[(x_1^2 + x_2^2)^2] > 0$

ifurcations: - qualitative change in phase portrait when a parameter $\mu$ passes some value

*. $\begin{cases} x_1' = \mu x_1 \\ x_2' = -x_2 \end{cases}$   $\mu$ parameter
$\mu < 0$ => stable node   $\mu = 0$ bifurcation point
$\mu > 0$ => saddle point

$x_1' = [\mu x_1 - x_2] - x_1(x_1^2 + x_2^2)$ ② → ? was not there initially.

$x_2' = [x_1 + \mu x_2] - x_2(x_1^2 + x_2)$

↑ Linearized system.

$\mu$ parameter

$-\infty < \mu < \infty$

eigenvalues of the linearized systems are $\lambda_{1,2} = \mu \pm i$.

$\mu < 0$ - stable focus

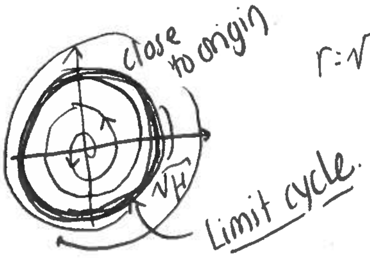
$\mu > 0$ - unstable focus.

$\mu = 0$ ???

multiplying first one by $x_1$, second one by $x_2$ => $r' = r(\mu - r^2)$, $\theta' = 1$

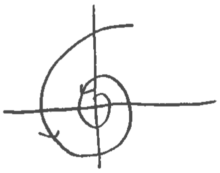
$\mu = 0 \Rightarrow r' = -r^3 < 0$ system is stable at $\mu = 0$. Can be seen as Liapunov function with $V = r^2$

$\boxed{\mu > 0}$

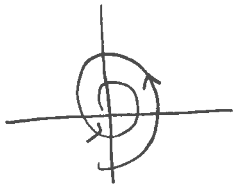


$r = \sqrt{\mu} \Rightarrow r' = 0$

$\boxed{\mu < 0}$



$\boxed{\mu = 0}$

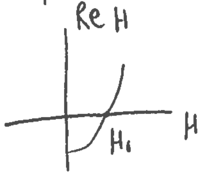


Hopf biturcation Th. 5.5.1 (A.P)

$\begin{cases} x_1' = f(x_1, x_2, \mu) \\ x_2' = f_2(x_1, x_2, \mu) \end{cases}$ has a fixed point in the origin for all values of a real parameter $\mu$.

1) Eigenvalues $\lambda_1(\mu), \lambda_2(\mu)$ are purely imaginary when $\mu = \mu_0$

) $\frac{d}{d\mu} \operatorname{Re}(\lambda_1(\mu))\Big|_{\mu=\mu_0} > 0$

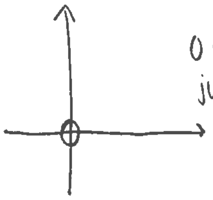


3) origin is asymptotically stable when $\mu = \mu_0$

hen: a) $\mu = \mu_0$ is a bifurcation point

b) for $\mu \in ]\mu_1, \mu_0[$ the system has a stable focus in the origin.

c) for $\mu > \mu_0$ origin is an unstable focus surrounded by a limit cycle with size nsing with $\mu$



$0 < \mu << 1$

just in a small neighbourhood of the origin.

$x_1' = \mu x_1 - 2x_2 - 2x_1(x_1^2 + x_2^2)^2$

$x_2' = 2x_1 + \mu x_2 - a x_2(x_1^2 + x_2^2)^2$

$\bar{x}' = \begin{bmatrix} \mu & -2 \\ 2 & \mu \end{bmatrix} \bar{x}$ $\lambda_{1,2} = \mu \pm i \cdot 2$

$\mu = 0$ might be a Hopf bifurcation point

$\lambda_{1,2}(0) = \pm i \cdot 2$

$\frac{d}{d\mu} \operatorname{Re}(\lambda) = \frac{d\mu}{d\mu} = 1 > 0.$

Asymptotic stability of the fixed point in the origin when $\mu = 0$?

$V(x_1, x_2) = x_1^2 + x_2^2$

$V' = 2x_1(-2x_2 - 2x_1(x_1^2 + x_2^2)^2) + 2x_2(2x_1 - 2x_2(x_1^2 + x_2^2)^2)$

$V' = -4x_1^2(x_1^2 + x_2^2)^2 - 2x_2^2(x_1^2 + x_2^2)^2 = -(x_1^2 + x_2^2)^2(2x_1^2 + x_2^2) \cdot 2 \Rightarrow$

n alternative way to show asymptotic stability for systems with linearization with center

lgorithm

) find a linearization $x' = Ax$ for $\mu = \mu_0 \Rightarrow \lambda_{1,2} = \pm i\beta$

) find a non-singular transformation matrix $M^{-1}AM = \begin{bmatrix} 0 & |\beta| \\ -|\beta| & 0 \end{bmatrix}$ - Jordan form.

$M = \begin{bmatrix} a_{11} & -\beta \\ a_{21} & 0 \end{bmatrix}$

) $\underline{x} = M\underline{y}$ => put into the non-linear system.

$y_1' = Y_1(y_1, y_2)$

$y_2' = Y_2(y_1, y_2)$ new formulation of the original system.

compute the index $I = |\beta|(Y^1_{111} + Y^1_{122} + Y^2_{112} + Y^2_{222}) + (Y^1_{11}Y^2_{11} - Y^1_{11}Y^1_{12} + Y^2_{11}Y^2_{12} + Y^2_{22}Y^2_{12} - Y^1_{22}Y^1_{12} - Y^1_{22}Y^2_{22})$ where $Y^i_{jk} = \dfrac{\partial^2 Y_i(0,0)}{\partial y_j \partial y_h}$ ; $Y^i_{jhl} = \dfrac{\partial^3 Y_i}{\partial y_j \partial y_h \partial y_l}$

If $I < 0$ => fixed point is asymp. stable.

<u>ex. 5.53</u> Show that the equation $x'' + (x^2 - \mu)x' + 2x + x^3 = 0$ has a Hopf bifurcation at $\mu = 0$.

$x_1' = x_2$

$x_2' = -(x_1^2 - \mu)x_2 - 2x_1 - x_1^3$ $\overset{\text{Linear-}}{\Rightarrow}$ ization $\bar{x}' = \begin{bmatrix} 0 & 1 \\ -2 & \mu \end{bmatrix} \bar{x}$ $M = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix} \Rightarrow M^{-1} \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} M = \begin{bmatrix} 0 & \sqrt{2} \\ -\sqrt{2} & 0 \end{bmatrix}$

New system: $y_1' = \sqrt{2}\,y_2$

$y_2' = -\sqrt{2}\,y_1 - y_1^2 y_2 - y_1^3/\sqrt{2}$

$I = -(Y^2_{112})\sqrt{2} = -2\sqrt{2}$

1915:ll poäng på tentan

maxp. 20

5: 17 p, 4: 14 p, 3: 10 p

×0.4

+2p. / inl. uppg.

×0.6

samma poängs. på projektet

samma skala på hela kursen

5 questions on exam
1. Linear system
2. Liapunov functions and stability
3 Periodic solutions
4 Hopf bifurcation
5 Gillespie method, chem react.

closed orbits must have a fixed point inside

<u>Brusselator</u> (<u>Prigozine</u>, Nikolis)

$A \to X$ A, B, C, D, E constants

$B + X \to Y + D$

$X + Y \to 3X$

$\to E$

$X' = a - (b+1)X + X^2Y$

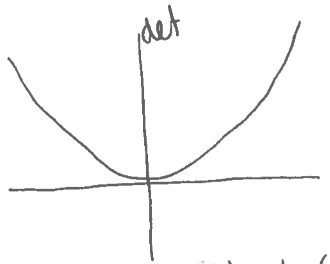
$Y' = bX - X^2Y$

a, b, concentrations of A, B, ..

1) Check fixed points <u>important</u>! If the system doesn't have fixed points → it has no periodic solutions.

Fixed point here $P = (a, b/a)$ ; $a, b > 0$.

2) Linearize the system around P. $J = \begin{bmatrix} 2XY - b - 1 & X^2 \\ b - 2XY & -X^2 \end{bmatrix} \Bigg|_{\substack{X = a \\ Y = b/a}} = \begin{bmatrix} b-1 & a^2 \\ -b & -a^2 \end{bmatrix}$ ;



$\det J(a, b/a) = a$ P stable for $a+1>b$

unstable for $a^2+1<b$.

$\lambda^2 - \lambda \,tr(J) + \det(J) = 0$

3) we suspect possible bifurcation at $a^2+1=b$. Check $Re(\lambda_{1,2})$

we fix a and consider

$x_i = X - a$

$x_2 = Y - b/a$

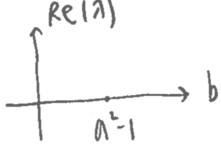
to get the fixed point in the origin, and gets:

Linearization

$$\begin{cases} x_1' = [(b-1)x_1 + a^2x_2] + 2a x_1x_2 + \frac{b}{a}x_1^2 + x_1^2x_2 \\ x_2' = [-bx_1 - a^2x_2] - 2ax_1x_2 - (\frac{b}{a})x_1^2 - x_1^2x_2 \end{cases}$$

i) $Re(\lambda_{1,2}) = \frac{1}{2}(b - a^2 - 1)$ for $(a-1)^2 < b < (a+1)^2$ (a fixed => fcn of b)

$\frac{d}{db}(\lambda(b)) = \frac{1}{2} > 0$ !!!



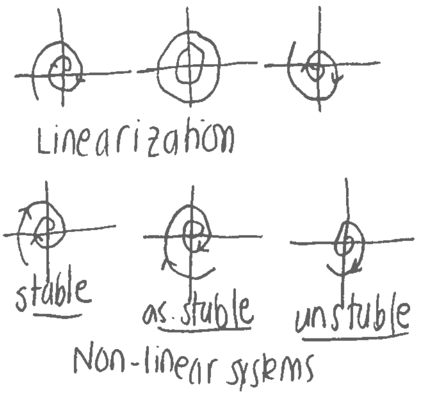
5) check asymptotical stability of the system for $b = a^2-1$. (Impossible to do by linearization! because the linearized system has a center)

has put b into the linearization

$$A = \begin{bmatrix} a^2 & a^2 \\ a^2+1 & -a^2 \end{bmatrix}; \quad M = \begin{bmatrix} a^2 & 0 \\ -a^2 & a \end{bmatrix} \quad M^{-1}AM = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$

$$\bar{x} = M\bar{y} \quad \begin{cases} y_1' = ay_2 + (1-a^2)ay_1^2 + 2a^2y_1y_2 - a^4y_1^3 + a^3y_1^2y_2 \\ y_2' = -ay_1 \end{cases}$$

$Y'_{111} \neq 0$ ; $Y'_{11}Y'_{12} \neq 0$ $\underline{I := -2a^5 - 4a^3 < 0}$ => as. stable.



Use Poincaré-Bendixson theorem to show that the system

$$\begin{cases} x' = -y + x(1-x^2-y^4) \\ y' = x + y(1-x^2-y^4) \end{cases}$$

has a periodic solution

Matrix to Linearized system $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

invariant and no fixed points => at least one closed orbit.

1) Idea: Find an invariant set. The simplest is to try $(x^2+y^2)'$ or $(x^2+ay^2)'$

$x'x + y'y = -yx + yx + (x^2+y^2)(1-x^2-y^4) = \frac{(r^2)'}{2}$ $(r^2)' = 2(x^2+y^2)(1-x^2-y^4)$

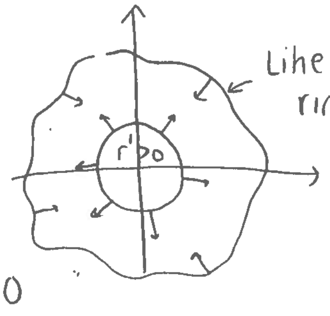
investigate where $(r^2)'>0$ and where $(r^2)'<0$

$(1-x^2-y^4)>0 \Rightarrow x^2+y^4<1$

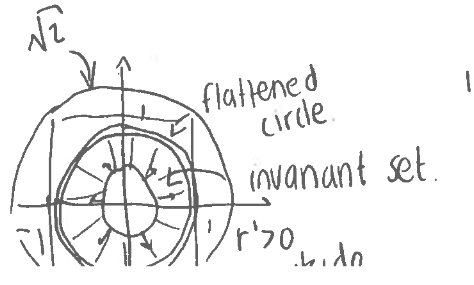
$(1-x^2-y^4)<0 \Rightarrow 1<x^2+y^4$

$x^2+y^4 < x^2+y^2 < 1$ for $r<1$

for $|y|<1$



Like to find a ringshaped domain positively invariant.



$1 < 2 < x^2 + y^4$ for $|x|>1$,

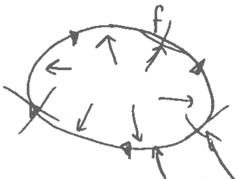
$|y|>1$. => $r'<0$

$0.5 < r < \sqrt{2}$ => invariant set $\begin{cases} 1<r \\ |x|<1, |y|<1 \end{cases}$

endixsons criterion

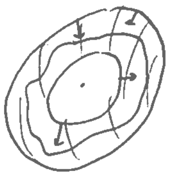
f div $\bar{f} \underset{<}{>} 0$ for the system $\bar{x}' = \bar{f}(\bar{x})$ in a simply connected domain $D$ in the plane, then the quation has no closed orbits in $D$.

ypical picture with closed orbits.



div $\bar{f} > 0$.
(no sinks, only positive sources in all points.)

Impossible!!



not simply connected!!!

## Linear systems

$\bar{x}' = A\bar{x}$ , $\bar{x}|_{t=0} = \bar{x}_0$ can be written $\bar{x} = e^{At}\bar{x}_0 = \varphi_t(x_0)$ evolution operator for the linear system

↗ for linear systems in any dimension.

- exact solution.

How to compute $e^{At}$ ?

1) Change of variables. => reduction to Jordan form.

2) Using the method by sylvester.

Both methods starts with finding eigenvalues. $\lambda^2 - tr(A) + det(A) = 0$.

$\lambda_{1,2}$) Re $\lambda_{1,2} > 0$ => unstable fixed point in the origin.
Re $\lambda_{1,2} < 0$ => stable fixed point in the origin.

2) Sylvester

$\underline{A = \lambda_1 Q_1 + \lambda_2 Q_2}$ ; $Q_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2}$ , $Q_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1}$

$Q_1 Q_2 = 0$
$Q_1^2 = Q_1$
$Q_2^2 = Q_2$

$e^{At} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!}$

↑ for different $\lambda_1 \neq \lambda_2$

$\exp(At) = e^{\lambda_1 t} Q_1 + e^{\lambda_2 t} Q_2$

## 12/5 Stability - Instability.

EF: stable fixed point $x^*$

For any neighbourhood $N$ of $x^*$, there is a neighbourhood $N_1$ such that, for any $\bar{x}_0 \in N_1$, $\varphi_t(\bar{x}_0) \in N$

TABILITY: $\forall$ neighbourhood $N \ni X_*$ $\exists$ neighbourhood $N_1 \ni X_*$ such that $\forall X_0 \in N_1 \Rightarrow \varphi_t(X_0) \subset N$

NSTABILITY: $\exists$ neighbourhood $N \ni X_*$ : $\forall N_1 \ni X_*$ $\exists$ $X_0 \in N$ such that $\varphi_t(X_0) \notin N$ ($\varphi_t(X_0)$ leaves $N$)

HEOREM $\bar{x}' = f(\bar{x})$ has a fixed point in the origin.

V is a real-valued continuous fcn such that.

a) The domain of V contains a ball $N = \{X : |X| \leq r\}$ for some $r > 0$.

b) There are points $\bar{X}_0$ arbitrarily close to the origin such that $V(X_0^i) > 0$

c) $V'(\bar{x}(t)) > 0$, $\bar{x} \neq 0$.

d) $\bar{V}(0) = 0$ $\Longrightarrow$ $X_*$ is unstable.

proof:

choose $\forall (X_0^i)$ (arbitrarily close to $\bar{x} = \bar{0}$) such that $V(\bar{X}_0) > 0$. We will show that $\varphi_t(X_0^i)$ has to leave N

$V'(\bar{x}) > 0$, $\bar{x} \neq \vec{0} \Rightarrow V'(\varphi_t(X_0^i)) > 0 \Rightarrow V(\varphi_t(X_0^i))$ is increased $\Rightarrow V(\varphi_t(X_0^i))$ can't go to zero $\Rightarrow V'(\varphi_t(X_0^i)) > K > 0 \Rightarrow \underset{t \to \infty}{\underrightarrow{V(\varphi_t(X_0^i))}} > V(X_0^i) + \underset{\text{constant}}{K}t$ $\infty$

ex 5.6 from chapter 5

consider $x'' + x'\sin((x')^2) + x = 0$

$$X_1 = X$$
$$X_1' = X_2$$
$$X_2' = -X_1 - X_2 \sin(X_2^2)$$

Show that the origin is a stable fixed point.

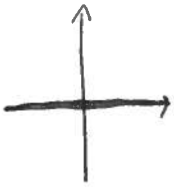
Try $V(X_1, X_2) = X_1^2 + X_2^2$ (or $aX_1^2 + bX_2^2$) $\Rightarrow V'(X_1, X_2) = \nabla V \begin{bmatrix} X_2 \\ -X_1 - X_2\sin(X_2^2) \end{bmatrix} = \begin{bmatrix} 2X_1 \\ 2X_2 \end{bmatrix} \begin{bmatrix} X_2 \\ -X_1 - X_2\sin(X_2^2) \end{bmatrix}$

$= 2X_1X_2 - 2X_1X_2 - 2X_2^2 \sin X_2^2 = -2X_2^2 \sin X_2^2$ ← Good guy.

Good terms: $-2X_1^2$ Bad $X_1, X_2$

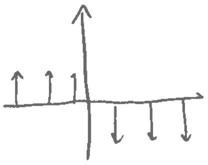
$-4X_2^2$

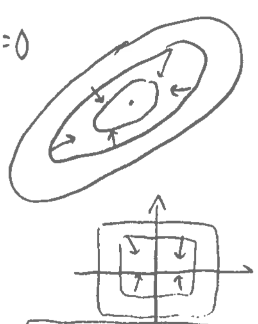
Good guys dominate the bad guys.

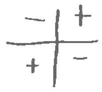


$X_1' = 0$

$X_2' = -X_1 = 0$ only for $X_1 = 0$

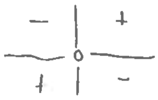






$V = X_1^2 + 2X_2^4$ doesn't work!

$aX_1^2 + 2bX_1X_2 + cX_2^2$ works



symmetric with respect to axes.

V is a weak Liapunov function. $\boxed{V' = 0, X_2 = 0}$

$X_1' = 0$

$X_2' = -X_1 = 0$ only for $X_1 = 0 \Rightarrow \vec{0}$ asymptotically stable!!

how that the origin is an asymptotically stable fixed point for the system

$x'' + (x')^3 + x^3 = 0 \qquad \begin{aligned} x_1' &= x_2 \\ x_2' &= -x_2^3 - x_1^3 \end{aligned}$

Try $V(x_1, x_2) = x_1^2 + x_2^2$

$v' = \underbrace{2x_1x_2}_{\text{indefinite}} - \underset{\overset{\wedge}{0}}{2x_2x_2^3} - \underbrace{2x_2x_1^3}_{\text{indefinite}} = 2x_2x_1(1 - \underbrace{x_1^2) - 2x_2^4}_{>0 \text{ if } |x_1|<1}$

Try $V(x_1,x_2) = x_1^4 + 2x_2^2$, $V' = 4x_1^3 + x_2 + 4x_2(-x_2^3 - x_1^3) = 4x_1^3x_2 - 4x_2x_1^3 - 4x_2^4 \leq 0$ $[=0 \text{ for } x_2 = 0]$

$x_2 = 0 \Rightarrow \begin{cases} x_1' = 0 \\ x_2' = -x_1^3 \neq 0, \ x_1 \neq 0. \end{cases}$

x. 5.6 investigate when $V(x_1,x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2 > 0$, $(x_1,x_2) \neq (0,0)$

$\underline{a>0}$: $V(x_1,x_2) = [x_1 \ \ x_2]\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \bar{x}^T A \bar{x} = \bar{y}^T \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \bar{y}$, $\bar{x} = M\bar{y}$

$= \lambda_1 y_1^2 + \lambda_2 y_2^2 > 0 \Leftrightarrow \lambda_1, \lambda_2 > 0$

M invertible, $\lambda_1, \lambda_2$ eigenvalues of A.

$\det \begin{bmatrix} a-\lambda & b \\ b & c-\lambda \end{bmatrix} = \lambda^2 - (a+c)\lambda + (ac - b^2) = 0 \Rightarrow \lambda_{1,2} = \frac{a+c}{2} \pm \sqrt{\frac{(a+c)^2}{4} - \underbrace{(ac-b^2)}_{<0,\ a>0 \atop c>0}} < 0.$

$\boxed{V' = ?}$ $\quad \begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - x_2 + (x_1 + 2x_2)(x_2^2 - 1) \end{aligned}$

Show that the origin is asymptotically stable using Liapunov fcn.

$V(x_1,x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$

$V' = 2ax_1x_2 + 2bx_2^2 - 2cx_1x_2 - 2cx_2^2 + 2cx_1x_2(x_2 - 1) + 4cx_2^2(x_2^2 - 1) - 2bx_1^2 - 2bx_1x_2 + 2bx_1^2 - 2bx_1x_2 +$

$+ \underset{\overset{\wedge}{0} \ |x_2|<1}{2bx_1^2(x_2^2 - 1)} + \underset{\overset{\wedge}{0},\ |x_2|<1}{4bx_1x_2(x_2^2 - 1)} =$

$= x_1^2(-2b - 2b(1 - x_2^2)) + x_2^2(2b - 2c + 4c(x_2^2 - 1)) + x_1x_2(2a - 2c + 2c(x_2^2 - 1)) + 4b\overset{<0}{(x_2^2 - 1)} <$

$< -Ax_1^2 + 2Bx_1x_2 - Cx_2^2 = \left\{ \begin{matrix} a=5, \ c=2 \\ b=1 \end{matrix} \right\} = x_1^2(-2 - 2(1 - x_2^2)) + x_1x_2(10 - 4 + 4(x_2^2 - 1) - 2) +$

$+ x_2^2(2 - 4 + 8(x_2^2 - 1)) \leq \{|x_2| < 1\} \leq x_1^2(-2) + x_2^2(-2) + x_1x_2(4)$